FINAL REPORT

PROJECT B-909

ANALYTICAL MECHANICAL MODEL FOR THE DESCRIPTION OF THE SLOSHING MOTION

By: Helmut F. Bauer, José Villanueva, and Shen Shu Chang

Contract No. NAS8-20203

Prepared for
George C. Marshall Space Flight Center
National Aeronautics and Space Administration
Huntsville, Alabama

December 31, 1966

[redacted]

GEORGIA INSTITUTE OF TECHNOLOGY
Atlanta, Georgia

60

FINAL REPORT

PROJECT B-909

ANALYTICAL MECHANICAL MODEL FOR THE DESCRIPTION OF THE SLOSHING MOTION

By: Helmut F. Bauer, José Villanueva, and Shen Shu Chang

Contract No. NAS8-20203

Prepared for
George C. Marshall Space Flight Center
National Aeronautics and Space Administration
Huntsville, Alabama

December 31, 1966

TABLE OF CONTENTS

LIST OF FIGURES

NOMENCLATURE

A_{mn}, B_{mn}, C_{mn}, D_{mn}:	Coefficients of the amplitude function and of the velocity potential function.
C_n:	$C_n = \epsilon_n \tanh\left(\epsilon_n \frac{h}{a}\right)$ characteristic length.
$\bar{c}_n$:	Damping coefficient of n^{th} mode.
F_x, F_y, F_z:	Components of the liquid force in x, y, z-axis directions respectively.
M_x, M_y, M_z:	Components of the liquid moment about x, y, z-axis.
$S(\bar{r}, \theta, \bar{z}, \bar{t})$:	Geometrical equation of the free surface.
T:	Kinetic energy.
V:	Potential energy.
Z:	Tank excitation amplitude (Mechanical Model).
Z^*:	Displacement of the sloshing mass at the wall.
Z_o:	Tank excitation amplitude (liquid).
ϵ:	Non-dimensional tank excitation amplitude.
a:	Tank radius.
h:	Liquid depth.
b:	Non-dimensional liquid depth.
m:	Total liquid mass.
m_n:	n^{th} sloshing mass.
m_o:	Non-sloshing mass.
2n-1:	Order of non-linear spring.
$\bar{r}$, θ, $\bar{z}$:	Tank fixed coordinate system.
r, θ, z:	Non-dimensional tank fixed coordinate system.
$\bar{t}$:	Time
u, v, w:	Fluid velocity components.

x_s, z_s:	Coordinates of the sloshing mass.
$\bar{a}_{mn}$:	Amplitude of the m, n^{th} liquid surface mode.
a_{mn}:	Non-dimensional amplitude of the m, n^{th} liquid surface mode.
$\bar{\alpha}_{mn}$:	Amplitude of the m, n^{th} component of the liquid velocity potential.
α_{mn}:	Non-dimensional amplitude of the m,n^{th} component of the liquid velocity potential.
ϵ_n:	n^{th} root of $J_1'(\epsilon_n) = 0$
$\bar{\lambda}_{mn}$:	Defined by $J_m'(\bar{\lambda}_{mn} a) = 0$.
λ_{mn}:	$\lambda_{mn} = \dfrac{\bar{\lambda}_{mn}}{\bar{\lambda}_{k\ell} \tanh \bar{\lambda}_{k\ell} h}$
$\bar{\zeta}$:	Free surface displacement above the mean level of liquid.
ζ:	Non-dimensional free surface displacement above the mean level of liquid.
$\bar{\rho}$:	Liquid density.
$\bar{\varphi}$:	Velocity potential.
φ:	Non-dimensional velocity potential.

SUMMARY

Liquid forces and moments due to longitudinal excitation of a rigid circular cylindrical container were obtained from nonlinear liquid theory. Through the anti-symmetric one-half-subharmonic response of the liquid these forces and moments will influence the lateral stability of a space vehicle.

A mechanical model describing this nonlinear liquid response has been derived and compared with liquid theory. The model consists of a mass with a moment of inertia which is rigidly connected to the container, and of independently oscillating masspoints capable of rolling on a guiding surface of paraboloidal form. Each of these masspoints is coupled with a nonlinear third order spring capable of moving up and down the longitudinal axis of the container.

1. INTRODUCTION

During the powered flight of a space vehicle longitudinal excitation can occur by dynamic coupling of structure and engine thrust through rough combustion or through thrust build-up or decay. It has been found that liquid in a container that is excited harmonically with a forcing frequency Ω will respond predominantly with a one-half subharmonic oscillation [1,2], i.e., half the frequency of the excitation.

Since during longitudinal excitation a one-half subharmonic response of the liquid in antisymmetric mode arises, and since this motion influences the lateral stability behavior of the space vehicle, an equivalent mechanical model should be derived that precisely describes this propellant motion.

For a circular cylindrical rigid container undergoing a longitudinal excitation, Dodge, Kana and Abramson [3] treated the nonlinear liquid problem, determined the free liquid surface elevations, and compared them with experimental results. The present work continues these investigations and determines in addition the pressure distributions, liquid forces, and moments. To describe the most important mode in application, namely the one-half subharmonic motion, the same equivalent nonlinear mechanical model shall be employed as has been derived for the translational case of excitation [4]. It consists of a mass having a moment of inertia, being rigidly attached to the container wall, and also having independently oscillating mass points for each vibration mode rolling on guiding surfaces of paraboloidal shape. Each of these mass points is coupled with a nonlinear spring capable of moving up and down the longitudinal axis of the container. The complete mechanical model is derived and compared with theoretical and experimental results. The unknown parameter of the model can thus be approximately obtained.

2. NON-LINEAR LIQUID THEORY

2.1 The Basic Equations

In this section the non-linear liquid wave equations valid for the case of a rigid container, are developed, neglecting capillary and surface tension effects on the wave surface. We assume that the liquid is inviscid and that its motion is irrotational, an assumption which is justified if the excitation frequency is not too close to one of the natural frequencies. The general method employed to solve the problem is similar to that used by Dodge, Kana and Abramson [3]. However, whereas Dodge, Kana and Abramson concentrate only on the liquid surface motion, we shall dedicate most of the effort on the computation of the velocity potential, the pressure distribution, the liquid force and the liquid moment.

The reference coordinate system employed (Figure 1) is fixed at the mean level of the liquid, and thus has the same motion as the tank. The base of the tank is excited with an axial motion given by $Z = Z_o \cos N\omega\bar{t}$, where ω is the frequency of the predominant liquid motions. Allowing N to assume various positive values specifies whether the predominant liquid motion is subharmonic, harmonic, or superharmonic.

Since the motion is irrotational, there exists a velocity potential $\bar{\varphi}$, such that

$$\vec{v} = - \text{grad } \bar{\varphi} \tag{1}$$

or in components ($\bar{u}$, $\bar{v}$, and $\bar{w}$ in the $\bar{r}$, θ, and $\bar{z}$ directions of the moving coordinate system respectively),

$$\bar{u} = -\frac{\partial\bar{\varphi}}{\partial\bar{r}}$$

$$\bar{v} = -\frac{1}{\bar{r}}\frac{\partial\bar{\varphi}}{\partial\theta}\text{, and} \tag{2}$$

$$\bar{w} = -\frac{\partial\bar{\varphi}}{\partial\bar{z}}$$

Since the fluid is incompressible, the potential $\bar{\varphi}$ must satisfy Laplace's equation

$$\nabla^2\bar{\varphi} = 0$$

i.e.

$$\frac{\partial^2\bar{\varphi}}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial\bar{\varphi}}{\partial\bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial^2\bar{\varphi}}{\partial\theta^2} + \frac{\partial^2\bar{\varphi}}{\partial\bar{z}^2} = 0 \tag{3}$$

Equation (3) must now be solved using the boundary conditions at the free surface and at the walls and bottom of the tank. The latter requires that the fluid velocity normal to the tank walls and bottom must be zero.

Consequently,

$$\bar{u} = -\frac{\partial\bar{\varphi}}{\partial\bar{r}} = 0 \quad \text{at} \quad \bar{r} = a \tag{4}$$

and

$$\bar{w} = -\frac{\partial\bar{\varphi}}{\partial\bar{z}} = 0 \quad \text{at} \quad \bar{z} = -h \tag{5}$$

The solution of equation (3) which also satisfies the boundary conditions (4) and (5) is given by

$$\bar{\varphi} = \bar{\alpha}_{oo} + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \bar{\alpha}_{mn}\, J_m(\bar{\lambda}_{mn}\bar{r})\cos m\theta \frac{\cosh \bar{\lambda}_{mn}(\bar{z}+h)}{\cosh \bar{\lambda}_{mnh}} \tag{6}$$

where $J_m(\bar{\lambda}_{mn}\,\bar{r})$ is the Bessel function of mth order and the first kind, and $\bar{\lambda}_{mn}$ is the nth root of the equation

$$\frac{d}{d\bar{r}}\left(J_m(\bar{\lambda}_{mn}\,\bar{r})\right)\Big|_{\bar{r}=a} = 0 \tag{7}$$

$\bar{\alpha}_{oo}$ and all the $\bar{\alpha}_{mn}$ are functions of time that have to be determined from the free surface boundary conditions.

The free surface boundary conditions are the dynamic and kinematic conditions. The first expresses that the pressure at the free surface must be constant and equal to the ullage pressure, and yields with Bernoulli's equation, the expression

$$\frac{\partial\bar{\varphi}}{\partial t}\Big|_{\bar{z}=\bar{\zeta}} - (g - N^2\omega^2 Z_o \cos N\omega\bar{t})\bar{\zeta} - \frac{1}{2}\left[\left(\frac{\partial\bar{\varphi}}{\partial\bar{r}}\right)^2 + \left(\frac{1}{\bar{r}}\frac{\partial\bar{\varphi}}{\partial\theta}\right)^2 + \left(\frac{\partial\bar{\varphi}}{\partial\bar{z}}\right)^2\right]_{\bar{z}=\bar{\zeta}} = 0 \tag{8}$$

where $\bar{\zeta}$ is the displacement of the free surface above its undisturbed position, $\bar{z} = 0$. (See Figure 1 for details).

Suppose $S\ (\bar{r},\ \theta,\ \bar{z},\ \bar{t})$ is the geometrical equation of the free surface. Then $S \equiv \bar{z} - \bar{\zeta}(\bar{r},\ \theta,\ \bar{t}) = 0$, and the kinematic boundary condition at the free surface may be written as (See Ref. 5),

$$\frac{DS}{D\bar{t}} = 0, \text{ or in expanded form}$$

$$\left[\frac{\partial\bar{\zeta}}{\partial\bar{t}} = \frac{\partial\bar{\varphi}}{\partial r}\cdot\frac{\partial\bar{\zeta}}{\partial\bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial\bar{\varphi}}{\partial\theta}\frac{\partial\bar{\zeta}}{\partial\theta} - \frac{\partial\bar{\varphi}}{\partial\bar{z}}\right]_{\text{at } \bar{z}=\bar{\zeta}} \tag{9}$$

In order to solve the problem one assumes that $\bar{\zeta}$ may be written as

$$\bar{\zeta} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \bar{a}_{mn}\, J_m(\bar{\lambda}_{mn}\,\bar{r}) \cos m\theta \tag{10}$$

where the $\bar{a}_{mn}$ are functions of time to be determined.

At this point we introduce a set of dimensionless variables (Ref. 3). If the sloshing under consideration is that which grows from the k, ℓth free (linear) mode, then it is reasonable to assume that $\bar{\alpha}_{k\ell}$ and $\bar{a}_{k\ell}$ are the predominant amplitudes in the series expansions of $\bar{\varphi}$ and $\bar{\zeta}$, and it is appropriate to nondimensionalize the variables by using the natural frequency and a characteristic length parameter of the corresponding free motion. Thus, let

$$\sigma = \frac{\omega}{\omega_{k\ell}} \quad \text{where} \quad \omega_{k\ell} = [\bar{\lambda}_{k\ell}\, g \tanh \bar{\lambda}_{k\ell} h]^{\frac{1}{2}}$$

$$t = \omega_{k\ell}\, \bar{t} \; ; \quad r = (\bar{\lambda}_{k\ell} \tanh \bar{\lambda}_{k\ell} h)\, \bar{r} \; ;$$

$$z = (\bar{\lambda}_{k\ell} \tanh \bar{\lambda}_{k\ell} h) \bar{z} \; ;$$

$$a_{mn} = (\bar{\lambda}_{k\ell} \tanh \bar{\lambda}_{k\ell} h) \bar{a}_{mn} \; ;$$

$$\zeta = (\bar{\lambda}_{k\ell} \tanh \bar{\lambda}_{k\ell} h) \bar{\zeta} \; ;$$

$$\alpha_{mn} = \frac{(\bar{\lambda}_{k\ell} \tanh \lambda_{k\ell} h)^2}{\omega_{k\ell}}\, \bar{\alpha}_{mn} \; ; \qquad \lambda_{mn} = \frac{\bar{\lambda}_{mn}}{\bar{\lambda}_{k\ell} \tanh \bar{\lambda}_{k\ell} h} \; ;$$

$$\epsilon = (\bar{\lambda}_{k\ell} \tanh \bar{\lambda}_{k\ell} h) Z_o \; ; \qquad \varphi = \frac{(\bar{\lambda}_{k\ell} \tanh \bar{\lambda}_{k\ell} h)^2}{\omega_{k\ell}}\, \bar{\varphi} \; ;$$

and

$$b = (\bar{\lambda}_{k\ell} \tanh \bar{\lambda}_{k\ell} h) h \tag{11}$$

Substituting these dimensionless variables into the boundary conditions, and expanding $\cosh \lambda_{mn}(\zeta + b)$ in equation (6) into a power series in ζ gives for the dynamic boundary condition (Equation (8)).

$$\ddot{\alpha}_{oo} + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left\{\left[\dot{\alpha}_{mn}F^{r}_{mn} - (1 - N^2\sigma^2\epsilon \cos N\sigma t)a_{mn}\right] J_m(\lambda_{mn}r)\cos m\theta\right\}$$

$$- \frac{1}{2} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} \cdot$$

$$\cdot \left\{\alpha_{mn}\alpha_{pq}\lambda_{mn}\lambda_{pq}J_{m+1}(\lambda_{mn}r)J_{p+1}(\lambda_{pq}r)\cos m\theta \cos p\theta F^{r}_{mn}F^{s}_{pq}\right.$$

$$+ \alpha_{mn}\alpha_{pq} \frac{mp}{r^2} J_m(\lambda_{mn}r)J_p(\lambda_{pq}r) \cos (m - p)\theta F^{r}_{mn}F^{s}_{pq}$$

$$+ \alpha_{mn}\alpha_{pq}\lambda_{mn}\lambda_{pq}J_m(\lambda_{mn}r)J_p(\lambda_{pq}r) \cos m\theta \cos p\theta G^{r}_{mn}G^{s}_{pq}$$

$$\left. - 2\alpha_{mn}\alpha_{pq} \frac{p\lambda_{mn}}{r} J_{m+1}(\lambda_{mn}r)J_p(\lambda_{pq}r)\cos m\theta \cos p\theta F^{r}_{mn}F^{s}_{pq}\right\} = 0 \qquad (12)$$

where the dots indicate differentiation with respect to time and

$$F^{\mu}_{\alpha\beta} = \frac{\cosh \lambda_{\alpha\beta}(\zeta + b)}{\cosh \lambda_{\alpha\beta}\zeta b} = \sum_{\mu=0}^{\infty} \left\{\frac{\lambda^{2\mu}_{\alpha\beta}\zeta^{2\mu}}{(2\mu)!} + \frac{\lambda^{2\mu+1}_{\alpha\beta}\ \zeta^{2\mu+1}}{(2\mu+1)!}\tanh \lambda_{\alpha\beta}b\right\} \qquad (13)$$

$$G^{\mu}_{\alpha\beta} = \frac{\sinh \lambda_{\alpha\beta}(\zeta + b)}{\cosh \lambda_{\alpha\beta}b} = \sum_{\mu=0}^{\infty} \left\{\frac{\lambda^{2\mu}_{\alpha\beta}\zeta^{2\mu}}{(2\mu)!} \tanh \lambda_{\alpha\beta}b + \frac{\lambda^{2\mu+1}_{\alpha\beta}\ \zeta^{2\mu+1}}{(2\mu + 1)!}\right\} \qquad (14)$$

and

$$\zeta = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{mn}J_m(\lambda_{mn}r) \cos m\theta$$

The kinematic boundary condition, Eq. (9), can be written as

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left\{\left[\dot{a}_{mn} + \lambda_{mn}\alpha_{mn}G^{r}_{mn}\right]J_m(\lambda_{mn}r)\cos m\theta\right\} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \cdot$$

$$\cdot\left\{\alpha_{mn}a_{pq}\lambda_{mn}\lambda_{pq}J_{m+1}(\lambda_{mn}r)J_{p+1}(\lambda_{pq}r)\cos m\theta\cos p\theta F^{r}_{mn}\right.$$

$$\alpha_{mn}a_{pq}\frac{mp}{r^2}J_m(\lambda_{mn}r)\,J_p(\lambda_{pq}r)\cos(m-p)\theta F^{r}_{mn}$$

$$-\alpha_{mn}a_{pq}\frac{m\lambda_{pq}}{r}J_m(\lambda_{mn}r)\,J_{p+1}(\lambda_{pq}r)\cos m\theta\cos p\theta F^{r}_{mn}$$

$$\left.-\alpha_{mn}a_{pq}\frac{p\lambda_{mn}}{r}J_{m+1}(\lambda_{mn}r)J_p(\lambda_{pq}r)\cos m\theta\cos p\theta F^{r}_{mn}\right\} \tag{15}$$

We can now solve equations (12) and (15) to give approximate values of the α_{mn} for special cases.

2.2 Antisymmetrical One-half Subharmonic Sloshing

Experimental evidence indicates that the $\frac{1}{2}$-subharmonic liquid motion is predominant. The mechanical model that shall be derived should therefore describe this motion. For this reason we investigate the large amplitude liquid motion whose period is twice that of the forcing function ($N = 2$ in equations (12) and (15)), corresponding to the first (lowest frequency) antisymmetrical sloshing mode. For this case α_{11} and a_{11} are the predominant amplitudes in the series expansions for $\bar{\zeta}$ and $\bar{\varphi}$, so that $k = \ell = 1$ in equations (11). Obviously, for practical reasons, we can only determine a finite number of terms in equations (12) and (15). Following the steps of Ref. 3 and restricting the solution to terms of order three and less, we need to consider only the following α's and a's: $\alpha_{00}^{(1)}$, $\alpha_{11}^{(1)}$, $\alpha_{01}^{(2)}$, $\alpha_{21}^{(2)}$, $a_{11}^{(1)}$, $a_{01}^{(2)}$, and $a_{21}^{(2)}$, where the superscript denotes the order of magnitude of the α's and a's relative to $\alpha_{11}\ a_{11}$.

Expanding equation (12) with this restriction in a Fourier series of cos mθ, one obtains:

$$
\Big[\dot{\alpha}_{00} + \{\dot{\alpha}_{01}J_0(\lambda_{01}r) - (1 - 4\sigma^2\epsilon\cos 2\sigma t)a_{01}J_0(\lambda_{01}r) + \tfrac{1}{2}\dot{\alpha}_{11}a_{11}J_1(\lambda_{11}r)J_1(\lambda_{11}r)\}
$$

$$
- \frac{1}{2}\{\frac{1}{2}\alpha_{11}^2\lambda_{11}^2J_2(\lambda_{11}r)J_2(\lambda_{11}r) + \alpha_{11}^2\frac{1}{r^2}J_1(\lambda_{11}r)J_1(\lambda_{11}r)
$$

$$
+ \frac{1}{2}\alpha_{11}^2\lambda_{11}^2J_1(\lambda_{11}r)J_1(\lambda_{11}r)\tanh^2\lambda_{11}b - \alpha_{11}^2\frac{\lambda_{11}}{r}J_2(\lambda_{11}r)J_1(\lambda_{11}r)\}\Big] +
$$

$$
\Big[\{\dot{\alpha}_{01}\lambda_{01}a_{11}J_1(\lambda_{11}r)\tanh\lambda_{01}b\, J_0(\lambda_{01}r)
$$

$$
+ \dot{\alpha}_{11}J_1(\lambda_{11}r)\,[1 + a_{01}\lambda_{11}J_0(\lambda_{01}r)\tanh\lambda_{11}b + \frac{1}{2}a_{21}\lambda_{11}J_2(\lambda_{21}r)\tanh\lambda_{11}b]
$$

$$
- a_{11}J_1(\lambda_{11}r)(1 - 4\sigma^2\epsilon\cos 2\sigma t) + \tfrac{1}{2}\dot{\alpha}_{21}a_{11}\lambda_{21}J_1(\lambda_{11}r)J_2(\lambda_{21}r)\tanh\lambda_{21}b\}
$$

$$
- \tfrac{1}{2}\{2\alpha_{01}\alpha_{11}\lambda_{01}\lambda_{11}J_1(\lambda_{01}r)J_2(\lambda_{11}r) + 2\alpha_{01}\alpha_{11}\lambda_{01}\lambda_{11}J_0(\lambda_{01}r)J_1(\lambda_{11}r)\tanh\lambda_{01}b\,\tanh\lambda_{11}b
$$

$$
- 2\alpha_0\alpha_{11}\frac{\lambda_{01}}{r}J_1(\lambda_{01}r)J_1(\lambda_{11}r) + \frac{3}{2}\alpha_{11}^2a_{11}\lambda_{11}^3J_2(\lambda_{11}r)J_2(\lambda_{11}r)J_1(\lambda_{11}r)\tanh\lambda_{11}b
$$

$$
+ 2\alpha_{11}^2a_{11}\lambda_{11}\frac{1}{r^2}J_1(\lambda_{11}r)J_1(\lambda_{11}r)J_1(\lambda_{11}r)\tanh\lambda_{11}b
$$

$$
+ \tfrac{3}{2}\alpha_{11}^2a_{11}\lambda_{11}^3J_1(\lambda_{11}r)J_1(\lambda_{11}r)J_1(\lambda_{11}r)\tanh\lambda_{11}b
$$

$$
- 3\alpha_{11}^2a_{11}\frac{\lambda_{11}^2}{r}J_2(\lambda_{11}r)J_1(\lambda_{11}r)J_1(\lambda_{11}r)\tanh\lambda_{11}b
$$

$$
+ \alpha_{11}\alpha_{21}\lambda_{11}\lambda_{21}J_2(\lambda_{11}r)J_3(\lambda_{21}r) + 4\,\alpha_{11}\alpha_{21}\frac{1}{r^2}J_1(\lambda_{11}r)J_2(\lambda_{21}r)
$$

$$
+ \alpha_{11}\alpha_{21}\lambda_{11}\lambda_{21}J_1(\lambda_{11}r)J_2(\lambda_{21}r)\tanh\lambda_{11}b\,\tanh\lambda_{21}b
$$

$$- 2\alpha_{11}\alpha_{21} \frac{\lambda_{11}}{r} J_2(\lambda_{11}r)J_2(\lambda_{21}r) - \alpha_{21}\alpha_{11} \frac{\lambda_{21}}{r} J_3(\lambda_{21}r)J_1(\lambda_{11}r)\}\Big] \cos \theta$$

$$+ \Big[\{\tfrac{1}{2}\dot{\alpha}_{11}a_{11}J_1(\lambda_{11}r)J_1(\lambda_{11}r)\tanh \lambda_{11}b + \dot{\alpha}_{21}J_2(\lambda_{21}r)$$

$$- a_{21}J_2(\lambda_{21}r)(1 - 4\sigma^2\epsilon \cos 2\sigma t)\}$$

$$- \frac{1}{2} \{\tfrac{1}{2}\alpha_{11}^2\lambda_{11}^2J_2(\lambda_{11}r)J_2(\lambda_{11}r) + \tfrac{1}{2}\alpha_{11}^2\lambda_{11}^2J_1(\lambda_{11}r)J_1(\lambda_{11}r) \tanh^2 \lambda_{11}b$$

$$- \alpha_{11}^2 \frac{\lambda_{11}}{r} J_2(\lambda_{11}r)J_1(\lambda_{11}r)\}\Big] \cos 2\theta$$

$$+ \Big[\{\tfrac{1}{2}\dot{\alpha}_{11}a_{21}J_1(\lambda_{11}r)J_2(\lambda_{21}r)\tanh \lambda_{11}b + \tfrac{1}{2}\dot{\alpha}_{21}a_{11}\lambda_{21}J_1(\lambda_{11}r)J_2(\lambda_{21}r) \tanh \lambda_{21}b\}$$

$$- \frac{1}{2} \{\tfrac{1}{2}\alpha_{11}^2\lambda_{11}^3a_{11}J_2(\lambda_{11}r)J_2(\lambda_{11}r)J_1(\lambda_{11}r)\tanh \lambda_{11}b$$

$$+ \tfrac{1}{2}\alpha_{11}^2\lambda_{11}^3a_{11}J_1(\lambda_{11}r)J_1(\lambda_{11}r)J_1(\lambda_{11}r) \tanh \lambda_{11}b \qquad (16)$$

$$- \alpha_{11}^2 \frac{\lambda_{11}^2}{r} a_{11}J_2(\lambda_{11}r)J_1(\lambda_{11}r)J_1(\lambda_{11}r) \tanh \lambda_{11}b$$

$$+ \alpha_{11}\alpha_{21}\lambda_{11}\lambda_{21}J_2(\lambda_{11}r)J_3(\lambda_{21}r)$$

$$+ \alpha_{11}\alpha_{21}\lambda_{11}\lambda_{21}J_1(\lambda_{11}r)J_2(\lambda_{21}r) \tanh \lambda_{11}b \tanh \lambda_{21}b$$

$$- 2\alpha_{11}\alpha_{21} \frac{\lambda_{11}}{r} J_2(\lambda_{11}r)J_2(\lambda_{21}r) - \alpha_{21}\alpha_{11} \frac{\lambda_{21}}{r} J_3(\lambda_{21}r)J_1(\lambda_{11}r)\}\Big] \cos 3\theta \approx 0$$

Since equation (16) must be valid for all values of θ, each term of the series must be independently equal to zero; thus one obtains three equations. The part of equation (16) which is independent of θ is

$$\dot{\alpha}_{00} + \{\dot{\alpha}_{01}J_0(\lambda_{01}r) - (1 - 4\sigma^2\epsilon\cos 2\sigma t)a_{01}J_0(\lambda_{01}r)$$

$$+ \frac{1}{2}\dot{\alpha}_{11}a_{11}J_1^2(\lambda_{11}r)\} - \frac{1}{2}\{\frac{1}{2}\alpha_{11}^2\lambda_{11}^2J_2^2(\lambda_{11}r) + \alpha_{11}^2\frac{1}{r^2}J_1^2(\lambda_{11}r)$$

$$+ \frac{1}{2}\alpha_{11}^2J_1^2(\lambda_{11}r) - \alpha_{11}^2\frac{\lambda_{11}}{r}J_2(\lambda_{11}r)J_1(\lambda_{11}r)\} = 0 \qquad (17)$$

where use has been made of the fact that $\lambda_{11}\tanh\lambda_{11}b = 1$.

We now expand equation (17) in a Bessel series of $J_0(\lambda_{0n}r)$, from which we obtain

$$\{\dot{\alpha}_{00} + 0.119345\,\dot{\alpha}_{11}a_{11} - (0.059673\,\lambda_{11}^2 + 0.059673)\alpha_{11}^2\}$$

$$+ \{\dot{\alpha}_{01} - (1 - 4\sigma^2\epsilon\cos 2\sigma t)a_{01} - 0.121482\dot{\alpha}_{11}a_{11} \qquad (18)$$

$$- \alpha_{11}^2(0.070796\lambda_{11}^2 - 0.060741)\}J_0(\lambda_{01}r) + \cdots = 0$$

Equation (18) now yields the following two ordinary non-linear differential equations:

$$\dot{\alpha}_{00} + 0.119345\dot{\alpha}_{11}a_{11} - (0.059673\,\lambda_{11}^2 + 0.059673)\alpha_{11}^2 = 0 \qquad (19)$$

for the part of equation (12) which is independent of both θ and r, and

$$\dot{\alpha}_{01} - (1 - 4\sigma^2\epsilon\cos 2\sigma t)a_{01} - 0.121482\dot{\alpha}_{11}a_{11}$$

$$- \alpha_{11}^2(0.070796\,\lambda_{11}^2 - 0.060741) = 0 \qquad (20)$$

for the part of equation (12) which is independent of θ, but dependent on ..

If we now expand the part of equation (16) which varies with $\cos\theta$ in a Bessel series of $J_1(\lambda_{1n}r)$ and that which varies with $\cos 2\theta$ in a Bessel series of $J_2(\lambda_{2n}r)$, we obtain respectively,

$$
\begin{aligned}
&[\dot{\alpha}_{11} - (1 - 4\sigma^2\epsilon\cos 2\sigma t)a_{11} + 0.104558\lambda_{11}^2\dot{\alpha}_{11}a_{11}^2 \\
&- 0.278821\lambda_{11}^2\dot{\alpha}_{11}^2 a_{11} - 0.165118\dot{\alpha}_{11}a_{01} + 0.198686\dot{\alpha}_{11}a_{21} \\
&- 0.165118\lambda_{01}\tanh\lambda_{01}b\dot{\alpha}_{01}a_{11} \\
&+ \alpha_{11}\alpha_{01}(0.165118\lambda_{01}\tanh\lambda_{01}b + 0.171812\lambda_{01}\lambda_{11}) \\
&+ 0.198686\lambda_{21}\tanh\lambda_{21}b\dot{\alpha}_{21}a_{11} \\
&- \alpha_{11}\alpha_{21}(0.198686\lambda_{21}\tanh\lambda_{21}b + 0.310343\lambda_{11}^2 \\
&- 0.022291\lambda_{11}\lambda_{21})]J_1(\lambda_{11}r)\cos\theta = 0
\end{aligned}
\tag{21}
$$

and

$$
\begin{aligned}
&[\dot{\alpha}_{21} - (1 - 4\sigma^2\epsilon\cos 2\sigma t)\, a_{21} + 0.350807\dot{\alpha}_{11}a_{11} \\
&- \alpha_{11}^2(0.175403 - 0.065931\lambda_{11}^2)]J_2(\lambda_{21}r)\cos 2\theta = 0
\end{aligned}
\tag{22}
$$

Similarly, from equation (15) one obtains the following three nonlinear ordinary differential equations:

$$
[\dot{a}_{01} + \alpha_{01}\lambda_{01}\tanh\lambda_{01}b - 0.263074\lambda_{11}^2\alpha_{11}a_{11}]J_o(\lambda_{01}r) = 0, \tag{23}
$$

$$[\mathring{a}_{11} + \alpha_{11} - 0.122515\lambda_{11}^2\alpha_{11}a_{11}^2 + 0.171812\lambda_{01}\lambda_{11}\alpha_{11}a_{01}$$

$$+ a_{11}\alpha_{01}(0.171812\lambda_{01}\lambda_{11} - 0.165118\lambda_{01}^2)$$

$$+ \alpha_{11}a_{21}(0.022291\lambda_{11}\lambda_{21} - 0.310343\lambda_{11}^2)$$

$$+ a_{11}\alpha_{11}(0.022291\lambda_{11}\lambda_{21} - 0.310343\lambda_{11}^2$$

$$+ 0.198686\lambda_{21}^2)]J_1(\lambda_{11}r) \cos \theta = 0 \tag{24}$$

and

$$[\mathring{a}_{21} + \alpha_{21}\lambda_{21} \tanh \lambda_{21}b + 0.482670\lambda_{11}^2\alpha_{11}a_{11}]J_2(\lambda_{21}r) \cos 2\theta = 0 \tag{25}$$

Equations (20) to (25) have to be solved for the α's and a's. Linearizing equations (21) and (24) and combining them, one obtains

$$\ddot{a}_{11} + (1 - 4\sigma^2\epsilon \cos 2\sigma t)a_{11} = 0$$

which is recognized as the Mathieu equation, a well-known result of linear theory.

Equations (20) through (25) are now combined to give three second-order ordinary differential equations in the time-varying amplitudes a_{11}, a_{01} and a_{21}. The results are (retaining 3rd order terms only),

$$\ddot{a}_{11} + (1 - 4\sigma^2\epsilon \cos 2\sigma t)a_{11}[1 + K_{11}a_{11}^2 + K_{01}a_{01} + K_{21}a_{21})$$

$$+ 0.034780\lambda_{11}^2\ddot{a}_{11}a_{11}^2 + k_{11}\mathring{a}_{11}^2a_{11} + 0.165118\ddot{a}_{01}a_{11}$$

$$- 0.198686\ddot{a}_{21}\, a_{11} + k_{01}\mathring{a}_{01}\mathring{a}_{11} - k_{21}\mathring{a}_{21}\mathring{a}_{11} = 0 \tag{26}$$

$$\ddot{a}_{01} + \lambda_{01} \tanh \lambda_{01}b\,(1 - 4\sigma^2\epsilon \cos 2\sigma t)a_{01} - \ddot{a}_{11}a_{11}(0.121482\lambda_{01} \tanh \lambda_{01}b -$$

$$- 0.263074\lambda_{11}^2) + \dot{a}_{11}^2[\lambda_{01}^2 \tanh \lambda_{01}b(0.070796\,\lambda_{11}^2 - 0.060741) +$$

$$+ 0.263074\,\lambda_{11}^2] = 0 \tag{27}$$

and

$$\ddot{a}_{21} + \lambda_{21} \tanh \lambda_{21}b(1 - 4\sigma^2\epsilon \cos 2\sigma t)a_{21} + \ddot{a}_{11}a_{11}(0.350807\lambda_{21} \tanh \lambda_{21}b$$

$$- 0.482670\,\lambda_{11}^2) + \dot{a}_{11}^2[\lambda_{21} \tanh \lambda_{21}b(0.175403 - 0.065931\,\lambda_{11}^2)$$

$$- 0.482670\,\lambda_{11}^2] = 0 \tag{28}$$

where

$$K_{11} = 0.122515 - \frac{0.045199 - 0.043438\lambda_{01}}{\tanh \lambda_{01}b} + \frac{0.010759\lambda_{11} + 0.09500\lambda_{21}}{\tanh \lambda_{21}b} -$$

$$- \frac{0.149793\lambda_{11}^2}{\lambda_{21}\tanh\lambda_{21}b},$$

$$K_{01} = 0.343624\,\lambda_{01}\lambda_{11}$$

$$K_{21} = 0.620686\,\lambda_{11}^2 - 0.044582\,\lambda_{11}\lambda_{21}$$

$$k_{11} = \frac{0.045199\lambda_{11}^3}{\tanh\lambda_{01}b} + \frac{0.149793\lambda_{11}^4}{\lambda_{21}\tanh\lambda_{21}b} - \frac{0.010759\lambda_{11}^3}{\tanh\lambda_{21}b}$$

$$k_{01} = 0.165118 + \frac{0.171812\lambda_{11}}{\tanh \lambda_{01}b}, \text{ and}$$

$$k_{21} = 0.198686 + \frac{0.310343\ \lambda_{11}^2}{\lambda_{21}\ \tanh\ \lambda_{21}b} - \frac{0.02229\lambda_{11}}{\tanh\ \lambda_{21}b} \tag{28-a}$$

Equations (27), (28), and (29) are solved assuming a solution of the form

$$a_{11} = A^{(1)}\ \sin\sigma t + A_{13}^{(3)}\ \sin 3\sigma t,$$

$$a_{01} = A_{00}^{(2)} + A_{02}^{(2)}\ \cos 2\sigma t\ ,\ \text{and}$$

$$a_{21} = A_{20}^{(2)} + A_{22}^{(2)}\ \sin 2\sigma t\ , \tag{29}$$

or of the form

$$a_{11} = B^{(1)}\ \cos\ \sigma t + B_{23}^{(3)}\ \sin 3\sigma t,$$

$$a_{01} = B_{00}^{(2)} + B_{02}^{(2)}\ \cos 2\sigma t\ ,\ \text{and}$$

$$a_{21} = B_{20}^{(2)} + B_{22}^{(2)}\ \cos 2\sigma t \tag{30}$$

where the superscript in parentheses indicates the order of the constants. However, the solution of the form of equations (30) is unstable (see Ref. 3) and can therefore not be observed experimentally.

The solution of equations (26) through (28) yields the following values for the A's of equation (30):

$$A_{00}^2 = a_{00}A^2 = -\ 0.066161\sigma^2\left[\frac{1.779184\sigma^2 - 0.925666 - 2\sigma^2\epsilon}{1.922058\sigma^2 - 1 + 8\sigma^4\epsilon^2}\right]A^2\ ,$$

$$A_{02}^2 = a_{02}A^2 = 0.066161\sigma^2\left[\frac{1 + 3.702665\sigma^2\epsilon}{1.922058\sigma^2 - 1 + 8\sigma^4\epsilon^2}\right]A^2\ ,$$

$$A_{20}^{2} = a_{20}A^{2} = 0.12066\sigma^{2}\left[\frac{1.215697\sigma^{2} - 0.504152 - 2\sigma^{2}\epsilon}{2.411370\sigma^{2} - 1 + 8\sigma^{4}\epsilon^{2}}\right] A^{2} ,$$

$$A_{22}^{2} = a_{22}A^{2} = - 0.120666\sigma^{2}\left[\frac{1 + 1.008304\sigma^{2}\epsilon}{2.411370\sigma^{2} - 1 + 8\sigma^{4}\epsilon^{2}}\right] A^{2} ,$$

$$A_{13}^{3} = - \frac{2\sigma^{2}\epsilon}{9\sigma^{2} - 1} A + a_{13}A^{3} = - \frac{2\sigma^{2}\epsilon}{9\sigma^{2} - 1} A + \frac{A^{3}}{9\sigma^{2} - 1} [0.123625$$

$$+ 0.741753\sigma^{2} - 0.039880\sigma^{2} - 1.430234\sigma^{2}\epsilon a_{00} + 0.357558a_{02} + 0.715117\sigma^{2}\epsilon a_{02}$$

$$+ 0.667166\sigma^{2}a_{02} + 1.093466\sigma^{2}\epsilon a_{20} - 0.273366a_{22} - 0.546733\sigma^{2}\epsilon a_{22}$$

$$- 0.762816\sigma^{2}a_{22}] ,$$

$$A^{2} = \left[\sigma^{2} - 1 - 2\sigma^{2}\epsilon - \frac{4\sigma^{4}\epsilon^{2}}{9\sigma^{2} - 1}\right] \Big/ [- 0.370877 - 0.989004\sigma^{2}\epsilon - 0.005100\sigma^{2}$$

$$+ 0.715117a_{00} + 1.430234\sigma^{2}\epsilon a_{00} - 0.357558a_{02} - 1.430234\sigma^{2}\epsilon a_{02}$$

$$+ 0.006694\sigma^{2}a_{02} - 0.546733a_{20} - 1.093466\sigma^{2}\epsilon a_{20} + 0.273366a_{22}$$

$$+ 1.093466\sigma^{2}\epsilon a_{22} + 0.032028\sigma^{2}a_{22} - 2\sigma^{2}\epsilon a_{13}], \qquad (31)$$

Thus, the non-dimensional displacement of the free surface is given by the expression:

$$\zeta = (A \operatorname{Sin}\sigma t + A_{13}^{3} \operatorname{Sin} 3\sigma t)\, J_{1} (\lambda_{11}r)\cos \theta +$$

$$+ (A_{00}^{2} + A_{02}^{2} \operatorname{Cos} 2\sigma t)\, J_{0} (\lambda_{01}r) + (A_{20}^{2} + A_{22}^{2} \operatorname{Cos} 2\sigma t)\, J_{2} (\lambda_{21}r) \operatorname{Cos} 2\theta \qquad (32)$$

The α's of equations (19) through (26) are assumed to be of the form:

$$\alpha_0 = M \text{ Sin } 2\sigma t \ ,$$

$$\alpha_{11} = C^{(1)} \text{ Cos } \sigma t + C_{13}^{(2)} \text{ Cos } 3\sigma t \ ,$$

$$\alpha_{01} = C_{00}^{(2)} + C_{02}^{(2)} \text{ Sin } 2\sigma t \ , \text{ and}$$

$$\alpha_{21} = C_{20}^{(2)} + C_{22}^{(2)} \text{ Sin } 2\sigma t \tag{33}$$

Substituting equations (33) into equations (19) through (26) and making use of equations (32) yield, (for the case $h \geq 2a$):

$$C_{00} = C_{20} = 0 \ ,$$

$$C \cong \frac{0.060741\sigma A}{0.070796\lambda_{11}^2 - 0.060741} + \sqrt{\left(\frac{0.060741\sigma A}{0.070796\lambda_{11}^2 - 0.060741}\right)^2 - \frac{2(A_{00} + 2\sigma^2 \epsilon A_{02})}{0.070796\lambda_{11}^2 - 0.060741}} \ ,$$

$$C_{02} = \frac{2\sigma A_{02} + 0.131535\lambda_{11}^2(AC)}{\lambda_{01} \tanh \lambda_{01} b} \ ,$$

$$C_{22} = \frac{2\sigma A_{22} + 0.241335\lambda_{11}^2(AC)}{\lambda_{21} \tanh \lambda_{21} b} \ ,$$

$$C_{13} = \frac{C^2(0.070796\lambda_{11}^2 - 0.060741) + 0.060741\sigma AC - 2\sigma C_{02} + A_{02} - 4\sigma^2 \epsilon A_{00}}{2[0.182223\sigma A - C(0.070796\lambda_{11}^2 - 0.060741)]} \ ,$$

and

$$M = \frac{1}{2\sigma}\left\{(0.059673\lambda_{11}^2 + 0.059673)\frac{C^2}{2} - \frac{0.119345}{2}\sigma AC\right\}. \tag{34}$$

Thus, the non-dimensional velocity potential may be written as

$$\begin{aligned}
\varphi \approx\ & M \sin 2\sigma t + (C \cos\sigma t + C_{13} \cos 3\sigma t)\, J_1(\lambda_{11} r) \cos\theta \frac{\cosh\lambda_{11}(z+b)}{\cosh\lambda_{11} b} \\
& + (C_{02} \sin 2\sigma t)\, J_0(\lambda_{01} r) \frac{\cosh\lambda_{01}(z+b)}{\cosh\lambda_{01} b} \\
& + (C_{22} \sin 2\sigma t)\, J_2(\lambda_{21} r) \cos 2\theta \frac{\cosh\lambda_{21}(z+b)}{\cosh\lambda_{21} b}
\end{aligned} \tag{35}$$

and the dimensional velocity potential, $\bar{\varphi}$, as obtained from equation (33) and (34) with the use of equations (11) yields the expression

$$\begin{aligned}
\bar{\varphi} = \sqrt{g}\,(\bar{\lambda}_{11} \tanh \bar{\lambda}_{11} h)^{-\frac{3}{2}} \{ & M \sin 2\sigma t \\
& + (C \cos \omega\bar{t} + C_{13} \cos 3\omega\bar{t})\, J_1(\bar{\lambda}_{11}\bar{r}) \cos\theta \frac{\cosh\bar{\lambda}_{11}(\bar{z}+h)}{\cosh\bar{\lambda}_{11} h} \\
& + (C_{02} \sin 2\omega\bar{t})\, J_0(\bar{\lambda}_{01}\bar{r}) \frac{\cosh\bar{\lambda}_{01}(\bar{z}+h)}{\cosh\bar{\lambda}_{01} h} \\
& + (C_{22} \sin 2\omega\bar{t})\, J_2(\bar{\lambda}_{21}\bar{r}) \cos 2\theta \frac{\cosh\bar{\lambda}_{21}(z+h)}{\cosh\bar{\lambda}_{21} h}
\end{aligned} \tag{35-a}$$

The pressure in the liquid at any position and at any time can be obtained from the unsteady Bernoulli equation, (8), and equation (35). It is

$$\frac{p}{\bar{\rho}} = \frac{p_o}{\bar{\rho}} + \frac{\partial\bar{\varphi}}{\partial\bar{t}} - (g - 4\omega^2 X_o \cos 2\omega\bar{t})\bar{z} - \frac{1}{2}\left[\left(\frac{\partial\bar{\varphi}}{\partial\bar{r}}\right)^2 + \left(\frac{1}{\bar{r}}\frac{\partial\bar{\varphi}}{\partial\theta}\right)^2 + \left(\frac{\partial\bar{\varphi}}{\partial\bar{z}}\right)^2\right] \qquad (36)$$

This yields the following expression for the pressure distribution

$$p = p_o - \bar{\rho}g(\bar{z} - \bar{\zeta} + S_2) - \bar{\rho}\sin\omega\bar{t}\left\{G\omega C\, J_1(\bar{\lambda}_{11}\bar{r})\cos\theta\,\frac{\cosh\bar{\lambda}_{11}(\bar{z}+h)}{\cosh\bar{\lambda}_{11}h} + S_1\right\}$$

$$- \bar{\rho}\sin 3\omega\bar{t}\left\{3G\omega C_{13}J_1(\bar{\lambda}_{11}r)\cos\theta\,\frac{\cosh\bar{\lambda}_{11}(\bar{z}+h)}{\cosh\bar{\lambda}_{11}h} + S_1\right\}$$

$$+ \rho\cos 2\omega\bar{t}\left\{2G\omega M + 2G\omega C_{02}\, J_o(\bar{\lambda}_{01}\bar{r})\,\frac{\cosh\bar{\lambda}_{01}(z+h)}{\cosh\bar{\lambda}_{01}h}\right.$$

$$\left. + 2G\omega\, C_{22}J_2(\bar{\lambda}_{21}\bar{r})\cos 2\theta\,\frac{\cosh\bar{\lambda}_{21}(z+h)}{\cosh\bar{\lambda}_{21}h} - S_2 + 4\omega^2 X_o(\bar{z} - \bar{\zeta})\right\} \qquad (37)$$

where

$$G = \sqrt{g}\,(\bar{\lambda}_{11}\tanh\bar{\lambda}_{11}h)^{-\frac{3}{2}},$$

$$S_1 = \frac{1}{2}G^2C\cos\theta\,\frac{\cosh\bar{\lambda}_{11}(\bar{z}+h)}{\cosh\bar{\lambda}_{11}h}\left[J_o(\bar{\lambda}_{11}\bar{r}) - \frac{J_1(\bar{\lambda}_{11}\bar{r})}{\bar{\lambda}_{11}\bar{r}}\right].$$

$$\left\{C_{22}\left[\frac{-2J_o(\bar{\lambda}_{21}\bar{r})}{\bar{\lambda}_{21}r} + \left(1 - \frac{4}{(\bar{\lambda}_{21}\bar{r})^2}\right)J_1(\bar{\lambda}_{21}\bar{r})\cos 2\theta\,\frac{\cosh\bar{\lambda}_{21}(\bar{z}+h)}{\cosh\bar{\lambda}_{21}h}\right.\right.$$

$$\left. - C_{02}\, J_1(\bar{\lambda}_{01}\bar{r})\,\frac{\cosh\bar{\lambda}_{01}(z+h)}{\cosh\bar{\lambda}_{01}h}\right\}$$

$$+ G^2CC_{22}\frac{1}{r^2}J_1(\bar{\lambda}_{11}\bar{r})J_2(\bar{\lambda}_{21}\bar{r})\sin\theta\sin 2\theta\,\frac{\cosh\bar{\lambda}_{11}(z+h)}{\cosh\bar{\lambda}_{11}h}\,\frac{\cosh\bar{\lambda}_{21}(z+h)}{\cosh\bar{\lambda}_{21}h}$$

$$+ \frac{G^2}{2} C\bar{\lambda}_{11} J_1(\bar{\lambda}_{11}\bar{r}) \cos\theta \frac{\sinh \bar{\lambda}_{11}(z+h)}{\cosh \bar{\lambda}_{11} h} \Big\{ C_{02}\bar{\lambda}_{01} J_0(\bar{\lambda}_{01}\bar{r}) \frac{\sinh \bar{\lambda}_{01}(\bar{z}+h)}{\cosh \bar{\lambda}_{01} h}$$

$$+ C_{22}\bar{\lambda}_{21} J_2(\bar{\lambda}_{21}\bar{r}) \cos 2\theta \frac{\sinh \bar{\lambda}_{21}(\bar{z}+h)}{\cosh \bar{\lambda}_{21} h} \Big\} ,$$

and

$$S_2 = \frac{1}{4} G^2 C^2 \left[J_o(\bar{\lambda}_{11}\bar{r}) - \frac{J_1(\bar{\lambda}_{11}\bar{r})}{\bar{\lambda}_{11}\bar{r}} \right]^2 \cos^2\theta \left[\frac{\cosh \bar{\lambda}_{11}(\bar{z}+h)}{\cosh \bar{\lambda}_{11} h} \right]^2$$

$$+ \frac{1}{4} G^2 C^2 \frac{1}{\bar{r}^2} J_1^2(\bar{\lambda}_{11}\bar{r}) \sin^2\theta \left[\frac{\cosh \bar{\lambda}_{11}(z+h)}{\cosh \bar{\lambda}_{11} h} \right]^2$$

$$+ \frac{1}{4} G^2 C^2 \bar{\lambda}_{11}^2 J_1^2(\bar{\lambda}_{11}\bar{r}) \cos^2\theta \left[\frac{\sinh \bar{\lambda}_{11}(z+h)}{\cosh \bar{\lambda}_{11} h} \right]^2$$

By integration of the appropriate pressure components, the liquid forces and moments can be obtained.

It is for the components of the liquid force:

$$F_x = \int_o^{2\pi} \int_{-h}^{\bar{\zeta}} p \Big|_{\bar{r}=a} \cos\theta \, R d\theta d\bar{z} \tag{38}$$

$$F_y = \int_o^{2\pi} \int_{-h}^{\bar{\zeta}} p \Big|_{\bar{r}=a} \sin\theta R d\theta d\bar{z} \equiv 0 \tag{39}$$

and

$$F_z = \int_o^{2\pi} \int_o^{a} p \Big|_{\bar{z}=-h} \bar{r} d\theta d\bar{r} \tag{40}$$

where:

$$p\big|_{\bar{r}=a} = p_o - \bar{\rho}[g(\bar{z} - \bar{\zeta}) + S_2\big|_{\bar{r}=a}]$$

$$- \bar{\rho} \sin \omega\bar{t}\left\{G\omega C J_1(\bar{\lambda}_{11}a)\cos\theta \frac{\cosh \bar{\lambda}_{11}(\bar{z}+h)}{\cosh \bar{\lambda}_{11}h} + S_1\big|_{\bar{r}=a}\right\}$$

$$- \bar{\rho} \sin 3\omega\bar{t}\left\{3G\omega C_{13} J_1(\bar{\lambda}_{11}a)\cos\theta \frac{\cosh \bar{\lambda}_{11}(\bar{z}+h)}{\cosh \bar{\lambda}_{11}h} + S_1\big|_{\bar{r}=a}\right\}$$

$$+ \rho \cos 2\omega\bar{t}\Big\{2G\omega M_2 + 2G\omega C_{02} J_0(\bar{\lambda}_{01}a) \frac{\cosh \bar{\lambda}_{01}(\bar{z}+h)}{\cosh \bar{\lambda}_{01}h}$$

$$+ 2G\omega C_{22} J_2(\bar{\lambda}_{21}a)\cos 2\theta \frac{\cosh \bar{\lambda}_{21}(\bar{z}+h)}{\cosh \bar{\lambda}_{21}h} - S_2\big|_{\bar{r}=a} + 4\omega^2 X_o(\bar{z} - \bar{\zeta})\Big\} \quad (41)$$

$$p\big|_{\bar{z}=-h} = p_o - \bar{\rho}[g(-h-\bar{\zeta}) + S_2\big|_{\bar{z}=-h}]$$

$$- \bar{\rho} \sin \omega t\left\{G\omega C J_1(\bar{\lambda}_{11}\bar{r})\cos\theta \frac{1}{\cosh \bar{\lambda}_{11}h} + S_1\big|_{\bar{z}=-h}\right\}$$

$$- \bar{\rho} \sin 3\omega\bar{t}\left\{3G\omega C_{13} J_1(\bar{\lambda}_{11}\bar{r})\cos\theta \frac{1}{\cosh \lambda_{11}h} + S_1\big|_{\bar{z}=-h}\right\}$$

$$+ \bar{\rho} \cos 2\omega\bar{t}\Big\{2G\omega M_2 + 2G\omega C_{02} J_o(\bar{\lambda}_{01}\bar{r}) \frac{1}{\cosh \bar{\lambda}_{01}h}$$

$$+ 2G\omega C_{22} J_2(\bar{\lambda}_{21}\bar{r})\cos 2\theta \frac{1}{\cosh \bar{\lambda}_{21}h} - S_2\big|_{\bar{z}=-h} + 4\omega^2 X_o(-h - \bar{\zeta})\Big\} \quad (42)$$

and

$$\bar{\zeta}\Big|_a \approx \frac{1}{\lambda_{11} \tanh \lambda_{11} h} A \sin \omega\bar{t}\, J_1(\bar{\lambda}_{11} a) \cos \theta \tag{43}$$

Integration of these expressions then yields:

$$\frac{F_x}{mg} = \left\{- \left(\frac{a}{h}\right)\left(\frac{\omega}{\omega_{11}}\right)(0.171644)\ C\right\} \sin \omega\bar{t} \tag{44}$$

$$\frac{F_y}{mg} = 0 \tag{45}$$

$$\frac{F_z}{mg} = 1 + - \left(\frac{a}{h}\right)\left[\frac{1}{(\tanh \bar{\lambda}_{11} h)^3} - \frac{1}{(\tanh \bar{\lambda}_{11} h)}\right] (0.016191)C^2$$

$$+ \left(\frac{a}{h}\right)\left(\frac{\omega}{\omega_{11}}\right) \frac{1}{\tanh (\bar{\lambda}_{11} h)} (1.086257)M_2 - 4 \left(\frac{\omega}{\omega_{11}}\right)^2 \epsilon \cos 2\omega\bar{t} \tag{46}$$

The liquid moment is determined from:

$$M_x = - \int_{\text{bottom}} y dF_z + \int_{\text{sides}} z dF_y = 0 \tag{47}$$

$$M_y = \int_{\text{bottom}} x dF_z + \int_{\text{sides}} z dF_x \tag{48}$$

$$M_z = 0 \tag{49}$$

where the variables inside the double integral are defined by eqs. (38), (39) and (40).

The integration yields, for the liquid moments,

$$M_x = 0 \tag{50}$$

$$M_z = 0 \tag{51}$$

$$\frac{M_y}{mga} = -\left\{\left(\frac{a}{h}\right)\left(\frac{\omega}{\omega_{11}}\right)\left[(2)\frac{\sqrt{1-\tanh^2 \bar{\lambda}_{11}h}}{\tanh \bar{\lambda}_{11}h} - \frac{1}{\tanh \bar{\lambda}_{11}h}\right](0.093224)C\right\}\sin\omega t \tag{52}$$

It can be noticed that the horizontal component of the liquid force remains stationary, i.e. it does not rotate, indicating that the coupling effect (present in the case of transverse excitation) does not appear in the case of longitudinal excitation.

The liquid moment has only one non-zero component, M_y.

3. NONLINEAR MECHANICAL SLOSH MODEL FOR LONGITUDINAL EXCITATION

The analytical mechanical model shall describe the antisymmetric liquid motion. The sloshing part of the liquid may be represented by a liquid volume of height, h_s, corresponding to the modal sloshing mass, m_s. The ratio of the first modal sloshing mass, m_s, to the total liquid mass, m, is considered (as in lateral sloshing) equal to the ratio of the height of the sloshing part of the liquid, h_s, to the total height of the liquid, h, i.e.,

$$\frac{h_s}{h} = \frac{m_s}{m} = \frac{2 \tanh\left(\epsilon_s \frac{h}{a}\right)}{\left(\epsilon_s \frac{h}{a}\right)\left(\epsilon_s^2 - 1\right)} \tag{53}$$

The same nonlinear mechanical model as in the case of lateral liquid sloshing shall be employed. Therefore, the same relation of displacement of the center of gravity of the sloshing part of the liquid is valid, i.e.

$$z_s = \frac{C_s}{2a} x_s^2 \tag{54}$$

where

$$C_s = \epsilon_s \tanh\left(\epsilon_s \frac{h}{a}\right) \tag{55}$$

It means that the sloshing mass m_s is constrained to move on a rotational paraboloid, which is subjected to longitudinal excitation

$$z(t) = Z_o \cos \Omega t \tag{56}$$

The mass point is furthermore coupled with a nonlinear spring capable of moving up and down the longitudinal axis of the container.

3.1 Equations of Motion

The equations of motion of the above described mechanical model, which should describe the antisymmetric liquid motion also for longitudinal excitation, are derived with the help of the Lagrange equation. This model, subjected to excitation in the z-direction and exhibiting a viscous damping, yields the expression for the kinetic energy in the form

$$T = \frac{m_s}{2} [\dot{x}_s^2 + (\dot{z}_s - \Omega Z_o \sin \Omega t)^2] \tag{57}$$

which, by the introduction of the equation of constraint,

$$f_s \equiv z_s - \frac{C_s}{2a} x_s^2 = 0 \tag{58}$$

yields the kinetic energy of the sloshing mass m_s

$$T = \frac{1}{2} m_s \left[\dot{x}_s^2 + \left(\frac{C_s}{a} x_s \dot{x}_s - \Omega Z_o \sin \Omega t\right)^2\right] \tag{59}$$

It may be mentioned here that the motion of the sloshing mass point is restricted to the parabola in the xz-plane.

The potential energy is given by

$$V = m_s g(z_s + Z_o \cos \Omega t) + \int_o^{x_s} k_s x_s^{2n-1} dx_s \tag{60}$$

which yields with the equation of constraint, (58), the expression

$$V = m_s g \frac{C_s}{2a} x_s^2 + m_s g\, Z_o \cos \Omega t + \frac{k_s}{2n} x_s^{2n} \tag{61}$$

The first term is the gravitational potential, the second term is due to change of the container location during the excitation, while the last term represents the energy stored in the nonlinear spring of order (2n-1).

The damping is assumed to be viscous, such that the mass point is subjected to a damping force proportional to its velocity relative to the paraboloid. The dissipation function therefore is given by the expression

$$D = \frac{\bar{c}_s}{2} (\dot{z}_s^2 + \dot{x}_s^2) \tag{62}$$

and, with

$$\bar{c}_s = 2m_s \omega_s \gamma_s \tag{63}$$

and the equation of constraint, (58), it becomes

$$D = m_s \omega_s \gamma_s \left[\dot{x}_s^2 + \frac{C_s^2}{a^2} \dot{x}_s^2 x_s^2 \right] \tag{64}$$

The introduction of these expressions into the Lagrange equation

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_s}\right] - \frac{\partial T}{\partial x_s} + \frac{\partial D}{\partial \dot{x}_s} + \frac{\partial V}{\partial x_s} = 0 \tag{65}$$

yields the equation of motion

$$\ddot{x}_s + 2\omega_s \gamma_s \dot{x}_s \left(1 + \frac{C_s^2}{a^2} x_s^2\right) + \frac{C_s^2}{a^2} (x_s \dot{x}_s^2 + x_s^2 \ddot{x}_s) + \omega_s^2 \left(1 + \frac{k_s}{m_s \omega_s^2} x_s^{2n-2}\right) x_s$$

$$= x_s \frac{C_s}{a} \Omega^2 Z_o \cos \Omega t \tag{66}$$

Introducing the dimensionless quantities

$$\xi_s = \frac{x_s}{a} \tag{67}$$

and

$$\alpha_s = \frac{k_s a^{2n-2}}{m_s \omega_s^2} \tag{68}$$

the above equation of motion yields

$$\ddot{\xi}_s + 2\omega_s \gamma_s \dot{\xi}_s (1 + C_s^2 \xi_s^2) + C_s^2 (\xi_s \dot{\xi}_s^2 + \xi_s^2 \ddot{\xi}_s) + \omega_s^2 (1 + \alpha_s \xi_s^{2n-2}) \xi_s$$

$$= \xi_s C_s \Omega^2 \left(\frac{Z_o}{a}\right) \cos \Omega t \tag{69}$$

This equation governs the motion of the sloshing mass due to longitudinal excitation, $Z_o \cos \Omega t$, of the system. It can be seen that the nonlinear ordinary time differential equation represents a Hill-type differential equation in which the coefficient is a function of time. Linearization of this equation yields the expression

$$\ddot{\xi}_s + 2\omega_s \gamma_s \dot{\xi}_s + \left[\omega_s^2 - C_s \Omega^2 \left(\frac{Z_o}{a}\right) \cos \Omega t\right] \xi_s = 0 \tag{70}$$

which represents a Mathieu-differential equation, as is also obtained from the linear theory.

3.2 Solution of the Equations of Motion

Of the various methods used in obtaining an approximate solution to nonlinear differential equations, the averaging procedure of Ritz seems to be the most appropriate one for this problem.

3.2.1 Undamped Response

If the motion is undamped, $\gamma_s = 0$, the differential equation (69) is

$$\ddot{\xi}_s + C_s^2(\xi_s\dot{\xi}_s^2 + \xi_s^2\ddot{\xi}_s) + \omega_s^2(1 + \alpha_s\xi_s^{2n-2})\xi_s = \xi_s C_s \Omega^2 \frac{Z_o}{a} \cos \Omega t \qquad (71)$$

An approximate solution for the steady state motion is given by

$$\bar{\xi}_s = A_s \sin \frac{\Omega}{2} t \qquad (72)$$

and, (with $\Omega t = \tau$), the Ritz condition,

$$\int_o^{4\pi} D[\bar{\xi}_s, \tau] \sin \frac{\tau}{2} d\tau = 0 \qquad (73)$$

yields

$$\int_o^{4\pi} \Big\{- \frac{\Omega^2}{4} A_s \sin \frac{\tau}{2} + C_s^2 \left(\frac{\Omega^2}{4} A_s^3 \sin \frac{\tau}{2} \cos^2 \frac{\tau}{2} - \frac{\Omega^2}{4} A_s^3 \sin^3 \frac{\tau}{2}\right)$$

$$+ \omega_s^2 A_s \sin \frac{\tau}{2}\left(1 + \alpha_s A_s^{2n-2} \sin^{2n-2} \frac{\tau}{2}\right)$$

$$- C_s \frac{Z_o}{a} \Omega^2 A_s \sin \frac{\tau}{2} \cos \tau\Big\} \sin \frac{\tau}{2} d\tau = 0 \qquad (74)$$

After evaluating the integrals with

$$\int_o^{2\pi} \sin^{2n} \frac{\tau}{2} d\tau = \frac{\pi(2n!)}{2^{2n-1} (n!)^2} \qquad (75)$$

and $r^2 = \frac{\Omega^2}{\omega_s^2}$, the expression for the frequency response function is

$$r^2 = \frac{1 + \frac{(2n!)}{2^{2n-1}(n!)^2}\,\alpha_s A_s^{2n-2}}{\frac{1}{4} + \frac{1}{8}\,C_s^2 A_s^2 - \frac{1}{2}\,C_s\,\frac{Z_o}{a}} \tag{76}$$

The above solution gives the stable portion of the sub-harmonic response curve, while $\bar{\xi}_s = A_s \cos\tau$ gives the unstable portion (see Ref. 3). For a cubic spring, i.e. $n = 2$, the frequency response function may be solved for A_s, and yields,

$$A_s = \frac{\frac{1}{4} - \frac{1}{r^2} - \frac{1}{2}\,C_s\,\frac{Z_o}{a}}{\frac{3}{4}\,\frac{\alpha_s}{r^2} - \frac{1}{8}\,C_s^2} \tag{77}$$

A discussion of the selection of a cubic spring is given in the next chapter. The non-dimensional spring constant, α_s, must be determined.

For purposes of comparison with experimental data, the calculations were performed for the test tank used by Dodge, Kana and Abramson [3] and for excitation amplitudes, $Z_o = 0.0516$ in. and $Z_o = 0.0258$ in. The tank has a radius of 2.86 in. and the fluid depth is 5.72 in. With the use of equation (77) the fluid amplitude at the wall of the container was determined and graphed versus r^2. Comparison with Dodge, Kana and Abramson's test results and theoretical results reveal that α_s should have the magnitude of two.

The response functions for this case are shown in Figures 3 and 10 using other values of α_s.

A better approximation can be achieved by considering a solution of the form

$$\bar{\xi}_s = A_s \sin \frac{\Omega}{2} t + B_s \sin \Omega t \tag{78}$$

With this assumption the Ritz conditions for the determination of the unknowns A_s and B_s are

$$\int_0^{4\pi} D[\bar{\xi}_s, \tau] \sin \frac{\tau}{2} \, d\tau = 0 \tag{79}$$

and

$$\int_0^{4\pi} D[\bar{\xi}, \tau] \sin \tau d\tau = 0 \tag{80}$$

which yield two simultaneous nonlinear algebraic equations in A_s and B_s. For $n = 2$, i.e. a cubic nonlinear spring, the resulting equations are

$$\frac{4-r^2}{4r^2} + C_s^2 \left(\frac{1}{8} A_s^2 - \frac{5}{8} B_s^2\right) + \frac{\alpha_s}{r^2}\left(\frac{3}{4} A_s^2 + \frac{3}{2} B_s^2\right) + \frac{1}{2} \frac{Z_o}{a} C_s = 0 \tag{81}$$

and

$$\left(\frac{1}{r^2} - 1\right) B_s + C_s^2 \left(- \frac{5}{8} A_s^2 B_s - \frac{1}{2} B_s^3\right) + \frac{\alpha_s}{r^2}\left(\frac{3}{2} A_s^2 B_s + \frac{3}{4} B_s^3\right) = 0 \tag{82}$$

These nonlinear equations for A_s and B_s as functions of r^2 were solved by the Newton-Raphson method. The resulting solutions indicated that the maximum magnitude of B_s is less than 1% of the value of A_s over the frequency range of interest $.9 \le r^2 \le 1.2$. From this we can conclude that the pure one-half subharmonic solution represents a very good approximation and definitely provides an acceptable degree of accuracy.

3.2.2 Damped Response

The damped response is readily obtained by the solution of the non-linear differential equation (69) which contains the damping term

$$2\omega_s \gamma_s \left(\dot{\xi}_s + C_s^2 \, \dot{\xi}_s \, \xi_s^2 \right) \tag{83}$$

and a third order spring (n = 2). This equation again is treated with the Ritz averaging method by assuming a solution of the form

$$\xi_s = A_s \sin \left(\frac{\Omega t}{2} + \psi_s \right) \tag{84}$$

where ψ_s is the phase of the motion relative to the excitation function. The Ritz conditions

$$\int_0^{4\pi} D[\bar{\xi}_s, \tau] \sin \frac{\tau}{2} \, d\tau = 0 \tag{85}$$

and

$$\int_0^{4\pi} D[\bar{\xi}_s, \tau] \cos \frac{\tau}{2} \, d\tau = 0 \tag{86}$$

yield the equations

$$- \frac{1}{4} r^2 - \gamma_s r \tan \psi_s - \frac{1}{4} \gamma_s r \, A_s^2 \, C_s^2 \tanh \psi_s + \frac{1}{16} C_s^2 A_s^2 r^2$$

$$- \frac{3}{16} C_s^2 A_s^2 r^2 + 1 + \frac{3}{4} \alpha_s A_s^2 + \frac{1}{2} C_s r^2 \left(\frac{Z_o}{a} \right) = 0 \tag{87}$$

and

$$- \frac{1}{4} r^2 + \gamma_s r \cos \psi_s + \frac{1}{4} \gamma_s r A_s^2 C_s^2 \cot \psi_s + \frac{1}{16} C_s^2 A_s^2 r^2$$

$$- \frac{3}{16} C_s^2 A_s^2 r^2 + 1 + \frac{3}{4} \alpha_s A_s^2 - \frac{1}{2} C_s r^2 \left(\frac{Z_o}{a}\right) = 0 \tag{88}$$

The damped frequency response is therefore determined from these equations, which yield for the phase angle,

$$\tan \psi_s = \frac{1 - \frac{1}{4} r^2 + \frac{3}{4} \alpha_s A_s^2 - \frac{1}{8} C_s^2 A_s^2 r^2 + \frac{1}{2} C_s \left(\frac{Z_o}{a}\right) r^2}{\gamma_s \left(1 + \frac{1}{4} A_s^2 C_s^2\right) r} \tag{89}$$

and, for the damped frequency response, the equation

$$\frac{1}{16} \left(9\alpha_s^2 + \frac{1}{4} C_s^4 r^4 - 3\alpha_s C_s^2 r^2 + \gamma_s^2 C_s^4 r^2\right) A_s^4$$

$$+ \left(\frac{3}{2} \alpha_s - \frac{1}{4} C_s^2 r^2 - \frac{3}{8} \alpha_s r^2 + \frac{1}{16} C_s^2 r^4 + \frac{1}{2} \gamma_s^2 C_s^2 r^2\right) A_s^2$$

$$+ \left[1 + \frac{1}{16} r^4 - \frac{1}{2} r^2 + \gamma_s^2 r^2 - \frac{1}{4} C_s^2 \left(\frac{Z_o}{a}\right)^2 r^4\right] = 0 \tag{90}$$

From this one obtains the expression:

$$\left[1 + \frac{1}{4} C_s^4 A_s^4 + C_s^2 A_s^2 - 4 C_s^2 \left(\frac{X_o}{a}\right)^2\right] r^4$$

$$+ \left(\gamma_s^2 C_s^4 A_s^4 - 3\alpha_s C_s^2 A_s^4 + 8\gamma_s^2 C_s^2 A_s^2 - 6\alpha_s A_s^2 - 4C_s^2 A_s^2 + 16\gamma_s^2 - 8\right) r^2$$

$$+ \left(9\alpha_s^2 A_s^4 + 24\alpha_s A_s^2 + 16\right) = 0 \tag{91}$$

For zero damping ($\gamma_s = 0$), the undamped response (Eq. 76) is obtained.

4. COMPARISON OF MECHANICAL MODEL AND LIQUID THEORY EQUATIONS

The liquid theory developed in Chapter 2 indicates that the wave amplitude may be assumed to be of the form (eq. 10)

$$\bar{\zeta} = \sum_{m=o}^{\infty} \sum_{n=1}^{\infty} \bar{a}_{mn} J_m \left(\bar{\lambda}_{mn} \frac{\bar{r}}{a}\right) \cos\, \theta$$

For the 1/2-sub-harmonic antisymmetrical mode, the dominant term is given by m = n = 1. Thus, retaining only the first term of the expansion,

$$\bar{\zeta} = \bar{a}_{11} J_1 (\bar{\lambda}_{11}\bar{r}) \cos\, \theta \tag{92}$$

where, for $\frac{h}{2a} \geq 1$, $\bar{a}_{11}$ is given by equation (27).

Taking only those terms of equation (27) which depend entirely on $\bar{a}_{11}$ and making the appropriate substitution from equation (11), one obtains (for $\frac{h}{2a} \geq 1$),

$$\ddot{\bar{a}}_{11} + (\omega_{11}^2 - 4\omega^2\, \bar{\lambda}_{11} X_o \cos 2\omega t)\, \bar{a}_{11} \cdot (1 + K_{11}\bar{\lambda}_{11}^2 \bar{a}_{11}^2)$$

$$+ 0.03478\, \bar{\lambda}_{11}^3 \ddot{\bar{a}}_{11} \bar{a}_{11}^{\,2} + k_{11}\, \bar{\lambda}_{11}^{\,3}\, \bar{a}_{11}^{\,2}\, \bar{a}_{11} = 0 \tag{93}$$

where K_{11} and k_{11} are constants given by equations (29-a).

Equation (93) may be non-dimensionalized in a manner similar to that used in the development of the equations of motion for the mechanical model. With

$$a\, \zeta_{11} = \bar{a}_{11}\,, \tag{94}$$

the result of substituting equation (94) into (93) is

$$\ddot{\zeta}_{11} + k_{11}(c_{11})^2 \zeta_{11}\dot{\zeta}_{11}^2 + 0.03478\,(c_{11})^2\, \zeta^2\ddot{\zeta} + \omega_{11}^2 \zeta_{11}$$

$$+ K_{11}\,(c_{11})^2\, \omega_{11}^2\, \rho_{11}^3 = 4\omega^2(c_{11})\,\frac{X_o}{a}\cos 2\,\omega t$$

where

$$c_{11} = a\bar{\lambda}_{11} \tanh\,(\bar{\lambda}_{11}h) = \epsilon_1 \tanh\left(\epsilon_1 \frac{h}{a}\right) = c_1 \tag{95}$$

as defined in equation (56). Also,use has been made of the fact that $\epsilon_n \overset{\Delta}{=} a\,\bar{\lambda}_{mn}$. The equation of motion for the undamped sloshing mass of the corresponding mechanical model is given by equation (69) as

$$\ddot{\xi}_1 + \sigma_1^2\xi_1\dot{\xi}_1^2 + c_1^2\xi_1^2\ddot{\xi}_1 + \omega_1^2\xi_1 + \alpha_1\omega_1^2\xi_1^{2n-1} = 4\omega^2(c_1)\,\frac{X_o}{a}\cos 2\omega t\;\; \xi_1 \tag{96}$$

where

$$\omega_1 = \omega_{11} = (\bar{\lambda}_{11}\; g \tanh \bar{\lambda}_{11}\; h)^{\frac{1}{2}}$$

Direct comparison of equations (95) and (96) indicates that n should be made equal to two, i.e. the non-linear spring should be a cubic spring.

Unfortunately, due to the complexity of the results from the non-linear liquid theory, it is not possible to determine analytically a proportionality factor between mass-displacement in the mechanical model and liquid amplitude, thus, α_1 cannot be determined except by direct comparison of numerical evaluations of liquid amplitudes, forces and moments as determined by experiments (where available), liquid theory, and by the mechanical model.

4.1 Derivation of the Mechanical Model of the Total Liquid System

The analytical mechanical analogy is designed in such a fashion that it describes the observed non-linear phenomena.

The liquid in the lower part of the container follows the motion like a rigid body, and is chosen to have a mass m_o and a moment of inertia I_o. The sloshing masses are denoted by m_n and the spring stiffnesses by k_n. The non-sloshing mass m_o is rigidly connected at a height h_o below the center of gravity of the quiescent liquid. To make the mechanical model equivalent to the fluid system, the sum of the modal masses must be equal to the total liquid mass. It is therefore

$$m = m_o + \sum_{n=1}^{\infty} m_n \tag{97}$$

Assuming that the sloshing masses are subjected to a damping force proportional to their velocity relative to the paraboloid, the dissipation function of the nth sloshing mass is

$$D_n(\dot{z}_n\ \dot{x}_n) = \frac{1}{2}\,\bar{c}_n\,(\dot{z}_n^2 + \dot{x}_n^2) \tag{98}$$

where $\bar{c}_n = 2m_n\omega_n\gamma_n$. The equations of motion of the mechanical model are now derived with the help of the Lagrange equations. For this reason one determines the kinetic and potential energy as well as the dissipation function of the system (see Figure 2). With x_n and z_n as the displacement of the nth sloshing mass, m_n, with respect to the container, with $x(t)$ and $z(t)$ as the displacements of the container and with φ the rotation of the container about the z-axis, the kinetic energy is given by:

$$T = \frac{m_o}{2}(\dot{x} - h_o\dot{\varphi})^2 + \frac{m_o}{2}(\dot{z})^2 + \frac{1}{2} I_o\dot{\varphi}^2$$

$$+ \frac{1}{2}\sum_{n=1}^{\infty} m_n\{[\dot{x}_n + \dot{x} + (h_n + x_n)\dot{\varphi}]^2 + [\dot{z} + \dot{z}_n - x_n\dot{\varphi}]^2\} \qquad (99)$$

The first three terms represent the kinetic energy of the non-sloshing mass, m_o, which is rigidly connected with the tank. The other term represents the kinetic energy of the sloshing masses, m_n. It may be remarked here that the pitching motion $\varphi(t)$ was assumed to exhibit small angles φ, such that $\cos\varphi \approx 1$ and $\sin\varphi \approx \varphi$.

The dissipation function is given by

$$D = \frac{1}{2}\sum_{n=1}^{\infty} \bar{c}_n(\dot{x}_n^2 + \dot{z}_n^2) \qquad (100)$$

The potential energy is composed of the lifting of the sloshing and non-sloshing masses and the energy stored in the springs. It is, for small values of φ,

$$V = \frac{m_o}{2} g h_o\varphi^2 - \frac{g}{2}\varphi^2\sum_{n=1}^{\infty} m_n(h_n + z_n) - g\varphi\sum_{n=1}^{\infty} m_n x_n + \frac{1}{4}\sum_{n=1}^{\infty} k_n x_n^4$$

$$+ \sum_{n=1}^{\infty} m_n g\, z_n + mgz \qquad (101)$$

The first term represents the potential energy due to the lifting of the non-sloshing mass, m_o, during a rotational $\varphi(t)$, while the second, third and fourth terms describe the same effect for the sloshing masses. The last term represents the accumulated energy in the springs. The coordinates x, z, φ, x_n, and

z_n are related by one equation of constraint, which expresses that the mass point m_n has to move on a paraboloid. The equation of constraint is

$$f \equiv z_n - \frac{C_n}{2a} x_n^2 = 0 \tag{102}$$

where C_n was found to be

$$C_n = \epsilon_n \tanh \left(\epsilon_n \frac{h}{a}\right) \tag{103}$$

Defining a Lagrange function L^*, as

$$L^* = L - \lambda f \tag{104}$$

L being the Lagrangian of the system, $L = T - V$, and λ a Lagrange multiplier, the equations of motion can be derived with the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \bar{q}_\nu}\right) - \frac{\partial L^*}{\partial \bar{q}_\nu} + \frac{\partial D}{\partial \bar{q}_\nu} = Q_\nu \tag{105}$$

where D is the dissipation function and Q_ν are the forces with respect to the coordinates q_ν. Another method for the derivation of the equations of motion from the Lagrange equation is based on the generalized coordinates q_ν which are, by elimination of the equation of constraint, made independent of each other. The kinetic energy and dissipation function in these generalized coordinates are:

$$\begin{aligned} T = {} & \frac{m_o}{2}(\dot{x} - h_o\dot{\varphi})^2 + \frac{m_o}{2}\dot{z}^2 + \frac{1}{2} I_o \dot{\varphi}^2 \\ & + \frac{1}{2}\sum_{n=1}^{\infty} m_n \left\{ \left[(\dot{x}_n + \dot{x}) + \left(h_n + \frac{C_n}{2a}x_n^2\right)\dot{\varphi}\right]^2 + \left[\dot{z} + 2\frac{C_n}{2a} x_n \dot{x}_n - x_n\dot{\varphi}\right]^2 \right\} \end{aligned} \tag{106}$$

and

$$D = \frac{1}{2} \sum_{n=1}^{\infty} \bar{c}_n \left[\dot{x}_n^2 + \left(\frac{C_n}{a} x_n \dot{x}_n \right)^2 \right] \tag{107}$$

The potential energy is given by

$$V = \frac{m_o}{2} g h_o \varphi^2 - \frac{g}{2} \varphi^2 \sum_{n=1}^{\infty} m_n \left(h_n + \frac{C_n}{2a} x_n^2 \right) - g\varphi \sum_{n=1}^{\infty} m_n x_n$$

$$+ \frac{1}{4} \sum_{n=1}^{\infty} k_n x_n^4 + \sum_{n=1}^{\infty} m_n g \frac{C_n}{2a} x_n^2 + mgz \tag{108}$$

The equations of motion are derived from the Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\nu} \right) + \frac{\partial \mathcal{D}}{\partial \dot{q}_\nu} - \frac{\partial T}{\partial q_\nu} + \frac{\partial V}{\partial q_\nu} = Q_\nu \tag{109}$$

where the generalized coordinates q_ν are x, z, φ and x_n; and $Q_x = -F_x$, $Q_z = -F_z$, $Q_\varphi = -M_y$, and $Q_{x_n} = 0$ are the generalized forces. The equations of motion are then:

$$m\ddot{z} + mg + \sum_{n=1}^{\infty} m_n \left[-x_n \ddot{\varphi} + 2 \frac{C_n}{2a} \left(\dot{x}_n^2 + x_n \ddot{x}_n \right) - \dot{x}_n \dot{\varphi} \right] = -F_z \tag{110}$$

$$m\ddot{x} - m_o h_o \ddot{\varphi} + \sum_{n=1}^{\infty} m_n \left[\ddot{x}_n + \left(\frac{C_n}{a} x_n \dot{x}_n \right) \dot{\varphi} + \left(h_n + \frac{C_n}{2a} x_n^2 \right) \ddot{\varphi} \right] = -F_x \tag{111}$$

$$I_o\ddot{\varphi} + m_o h_o \dot{x} \quad - m_o h_o^2 \varphi + \sum_{n=1}^{\infty} m_n \Big\{ \Big[\ddot{x}_n + \ddot{x} + \Big(\frac{C_n}{a} x_n \dot{x}_n\Big)\dot{\varphi} +$$

$$\Big(h_n + \frac{C_n}{2a} x_n^2\Big)\ddot{\varphi}\Big]\Big(h_n + \frac{C_n}{2a} x_n^2\Big) + \Big[\dot{x}_n + \dot{x} + \Big(h_n + \frac{C_n}{2a} x_n^2\Big)\dot{\varphi}\Big]\Big(\frac{C_n}{a} x_n \dot{x}_n\Big)$$

$$- \Big[\ddot{x} + \frac{C_n}{a} \dot{x}_n^2 + \frac{C_n}{a} x_n \ddot{x}_n - \dot{x}_a \dot{\varphi} - x_n \ddot{\varphi}\Big] x_n - \Big(\dot{z} + \frac{C_n}{a} x_n \dot{x}_n - x_n \dot{\varphi}\Big)\dot{x}_n \Big\}$$

$$+ m_o g h_o \varphi - g\varphi \sum_{n=1}^{\infty} m_n \Big(h_n + \frac{C_n}{2a} x_n^2\Big) - g \sum_{n=1}^{\infty} m_n x_n = -M_y \qquad (112)$$

and

$$m_n \Big\{ \Big[\ddot{x}_n + \ddot{x} + \Big(\frac{C_n}{a} x_n \dot{x}_n\Big)\dot{\varphi} + \Big(h_n + \frac{C_n}{2a} x_n^2\Big)\ddot{\varphi}\Big]$$

$$+ \Big[\ddot{x} + \frac{C_n}{a} \dot{x}_n^2 + \frac{C_n}{a} x_n \ddot{x}_n - \dot{x}_n \dot{\varphi} - x_n \ddot{\varphi}\Big] \Big(+ \frac{C_n}{a} x_n\Big)$$

$$+ \Big[\dot{z} + \frac{C_n}{a} x_n \dot{x}_n - x_n \dot{\varphi}\Big]\Big(+ \frac{C_n}{a} \dot{x}_n\Big) \Big\}$$

$$+ m_n \Big\{ \Big[\dot{x}_n + \dot{x} + \Big(h_n + \frac{C_n}{2a} x_n^2\Big)\dot{\varphi}\Big]\Big(- \frac{C_n}{a} \dot{\varphi} x_n\Big)$$

$$+ \Big[\dot{z} + \frac{C_n}{a} x_n \dot{x}_n - x_n \dot{\varphi}\Big]\Big(\dot{\varphi} - \frac{C_n}{a} \dot{x}_n\Big) \Big\} + \bar{c}_n \Big[\dot{x}_n + \frac{C_n^2}{a^2} x_n^2 \dot{x}_n\Big] - \frac{g}{2} \varphi^2 \cdot$$

$$\cdot\, m_n \Big(\frac{C_n}{a} x_n\Big) - g\varphi m_n + k_n x_n^3 + m_n g \frac{C_n}{a} x_n = 0 \qquad (113)$$

If we now let $x(t) \equiv \varphi(t) \equiv 0$ as is the case, we obtain:

$$- F_z = mg + m\ddot{z} + \sum_{n=1}^{\infty} \frac{m_n C_n}{a} (\dot{x}_n^2 + x_n \ddot{x}_n) \qquad (114)$$

$$- F_x = + \sum_{n=1}^{\infty} m_n \ddot{x}_n \tag{115}$$

$$- M_y = \sum_{n=1}^{\infty} m_n \left\{ -\dot{z}\dot{x}_n - \ddot{z}x_n + h_n \ddot{x}_n - \frac{C_n}{2a} x_n^2 \ddot{x}_n - \frac{C_n}{a} x_n \dot{x}_n^2 - g x_n \right\} \tag{116}$$

and

$$\ddot{x}_n + 2\omega_n \gamma_n \dot{x}_n \left(1 + \frac{C_n^2}{a^2} x_n^2\right) + \frac{C_n^2}{a^2} (x_n \dot{x}_n^2 + x_n^2 \ddot{x}_n)$$

$$+ \omega_n^2 \left(1 + \frac{k_n}{m_n \omega_n^2} x_n^2\right) x_n = x_n \frac{C_n}{a} \Omega^2 Z_o \cos \Omega t \qquad n = 1, 2, \cdots \tag{117}$$

The first three equations give the forces and moments which the mechanical model exerts on the container, whereas the last equation is the equation of motion of the sloshing mass, m_n, in the undamped case.

Considering the forces produced by the first sloshing mass, m_1, substituting the mechanical values, and letting

$$x_n = A\, a \sin \frac{\Omega}{2} t \tag{118}$$

and

$$z = Z_o \cos \Omega\, t \tag{119}$$

we obtain,

$$\left|\frac{F_z}{mg}\right| = 1 + \left|\frac{\epsilon_1}{(\epsilon_1^2 - 1)} \cdot \frac{\tanh^3 [\epsilon, \frac{h}{a}]}{\frac{h}{a}} A^2 \right| r^2 - \left|\frac{Z_o}{a}\, \epsilon_1 \tanh\left(\frac{\epsilon_1 h}{a}\right)\right| r^2 \tag{120}$$

$$\left|\frac{F_x}{mg}\right| = \frac{\tanh^2(\epsilon_1 \frac{h}{a})}{2(\frac{h}{a})(\epsilon_1^2-1)} A\ r^2, \tag{121}$$

and

$$\left|\frac{M_y}{mag}\right| = \frac{\tanh^2(\epsilon_1 \frac{h}{a})\ Ar^2}{\frac{h}{a}\ (\epsilon_1^2-1)} \left[\frac{1}{16}\ \epsilon_1\ \tanh\ [\epsilon_1\ \frac{h}{a}]\ A^2 - \frac{\tanh\ (\epsilon_1\ \frac{h}{a})}{\epsilon_1\ (\epsilon_1^2-1)}\right.$$

$$\left. - \frac{2}{\epsilon_1\ \tanh(\epsilon_1\ \frac{h}{a})} \cdot \frac{1}{r^2} - \frac{1}{2}\left(\frac{Z_o}{a}\right) \right] * \tag{122}$$

These values are plotted in Figures 3 through 10, and are compared to the results of the liquid theory from Section 2.

*Note:

$\left|\frac{M_y}{mga}\right|$ of equation (122) is given with respect to the origin at the center of mass of the undisturbed liquid.

5. NUMERICAL RESULTS AND CONCLUSIONS

In order to take advantage of the experimental results of Dodge, Kana, and Abramson [3], the numerical evaluations were performed for a tank which was 5.72 inches in diameter and with $h/a = 2$. Two different excitation amplitudes were used in the computations: $Z_o = 0.0258$ inches and $Z_o = 0.0516$ inches.

The non-dimensional liquid amplitude Z^* as determined from the liquid theory was evaluated at a point $\bar{r} = 2.49$ inches and $\theta = 0$ by using the equation

$$Z^* = \frac{1}{2}\left[\zeta\left(\frac{\bar{r}}{a} = 0.837, \theta = 0, \sigma t = \frac{\pi}{2}\right) - \zeta\left(\frac{\bar{r}}{a} = 0.837, \theta = 0, \sigma t = \frac{3\pi}{2}\right)\right] \quad (123)$$

where $\zeta(r, \theta, t)$ is given by equation (32).

For the mechanical model, Z^* is given by

$$Z^* = \frac{2\epsilon_1 \tanh\left(\epsilon_1 \frac{h}{a}\right)}{\epsilon_1^2 - 1} \cdot J_1(0.837\ \epsilon_1) \cdot \xi_1\left(\frac{\Omega t}{2} = \frac{\pi}{2}\right) \quad (124)$$

since the relationship between the displacement of the sloshing mass and the liquid amplitude is given by equation (2.7) of Ref. 4 as:

$$\xi_1 = \frac{(\epsilon_1^2 - 1)\ Z}{2\epsilon_1 \tanh\left(\epsilon_1 \frac{h}{a}\right)}$$

The results of equations (123) and (124) are shown plotted in Figs. 3 and 4. It is evident that liquid theory and the mechanical model agree fairly well with the experimental values for small values of Z^*; however, whereas the liquid theory predict much smaller amplitudes than those observed in experiments, the mechanical model not only yields values which are in agreement with the experimental results but also shows the inflection points as in the experimental results.

Comparison of the liquid forces and moments must be made on a direct basis between the liquid theory and the mechanical model, since there are no experimental values available. We now see that a value of $\alpha = 2$ is best suited. Again there is close agreement between the liquid theory and the mechanical model in the range for which Z^* was small. However, in the frequency range for which the liquid theory predicted smaller amplitudes than the experiments showed, one finds that the mechanical model predicts larger values for both the liquid forces and the liquid moment. As would be expected, the transverse component of the liquid force, shown in Figs. 5 and 6 yields the same type of inflection points as those given by the non-dimensional liquid amplitude Z^*.

Summarizing, one concludes that the mechanical model should describe the non-linear sloshing phenomena accurately enough for most engineering applications.

In fact, it yields better results for the free surface elevation than the non-linear liquid theory.

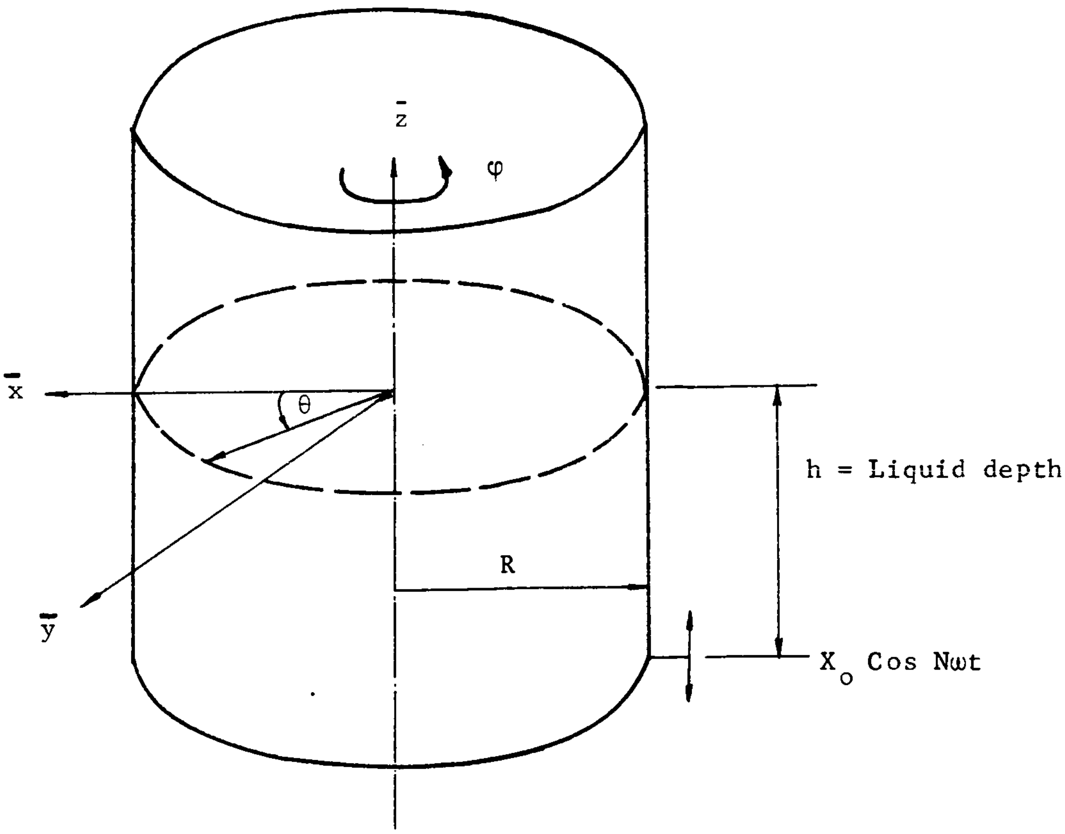


Figure 1. Cylindrical Tank and Coordinate System

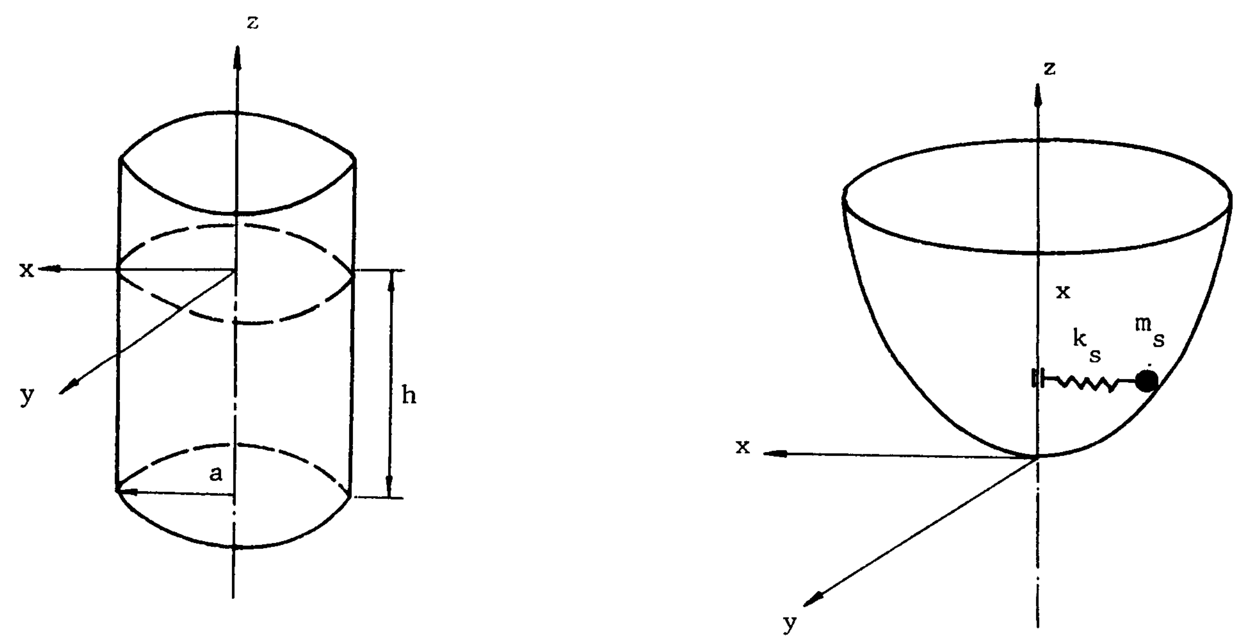


Figure 2. Mechanical Model and Coordinate System

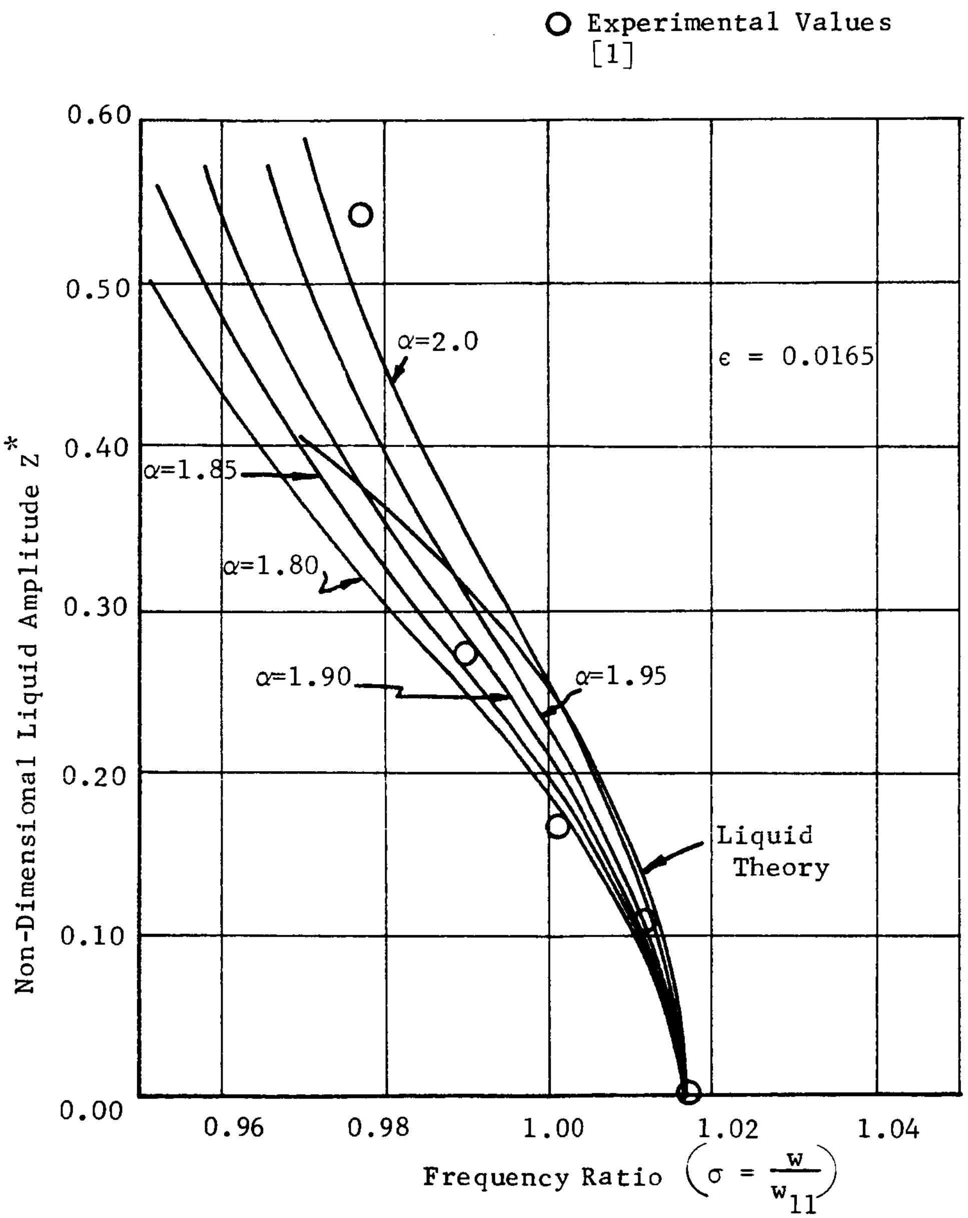


Figure 3. Non-Dimensional Liquid Amplitude (ϵ = 0.0165)

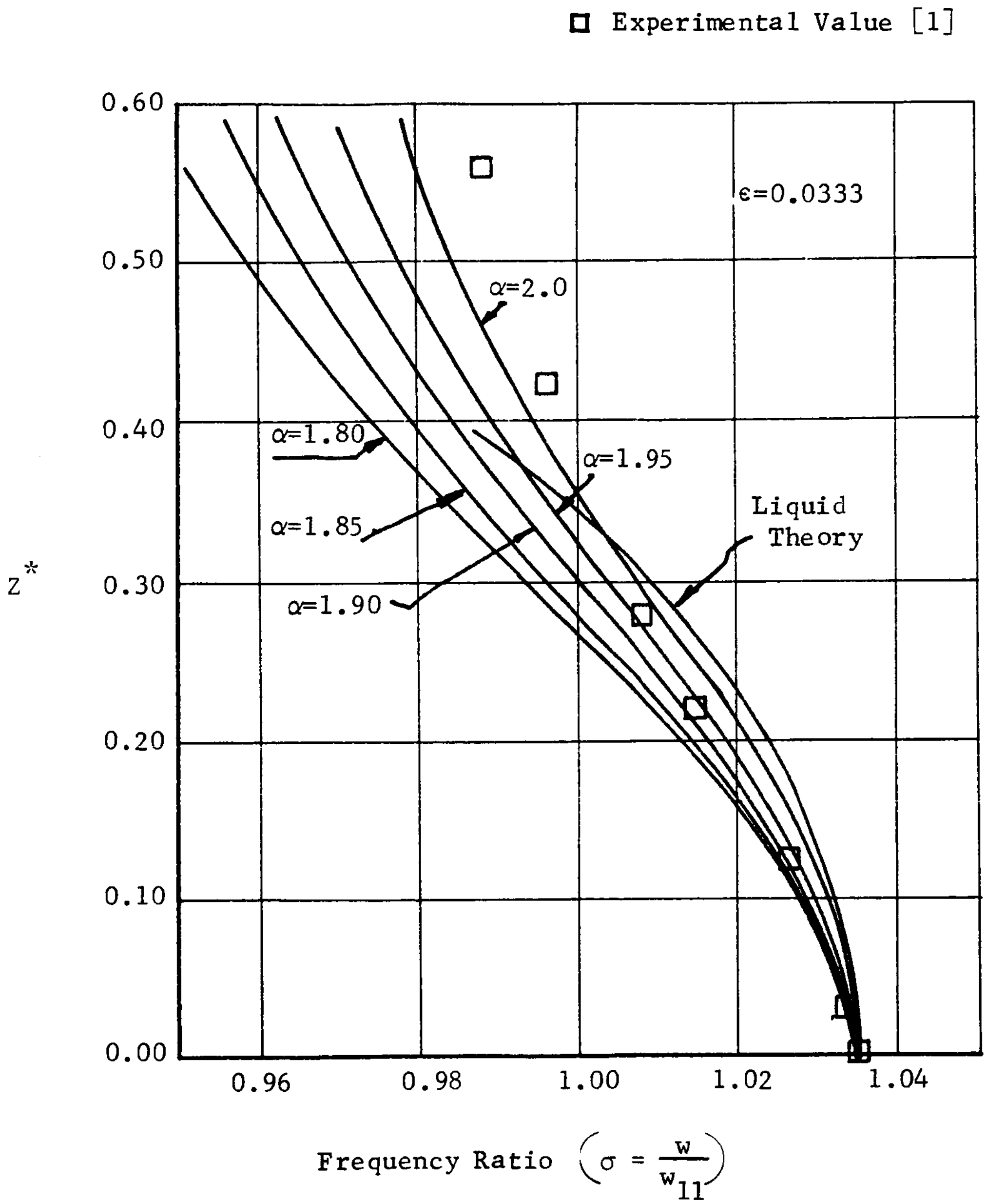


Figure 4. Non-Dimensional Liquid Amplitude (ϵ = 0.0333)

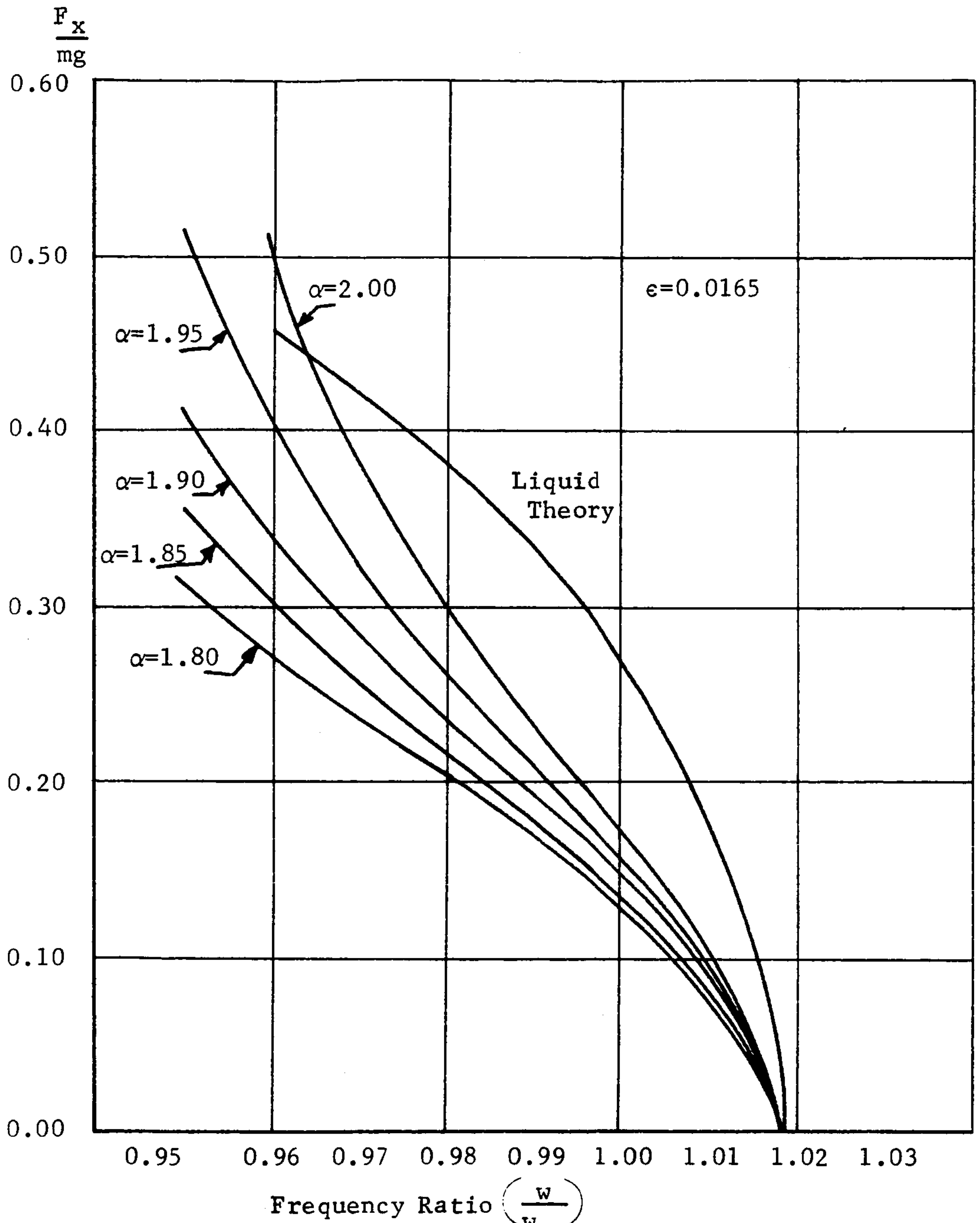


Figure 5. Liquid Force, Transverse Component ($\epsilon = 0.0165$)

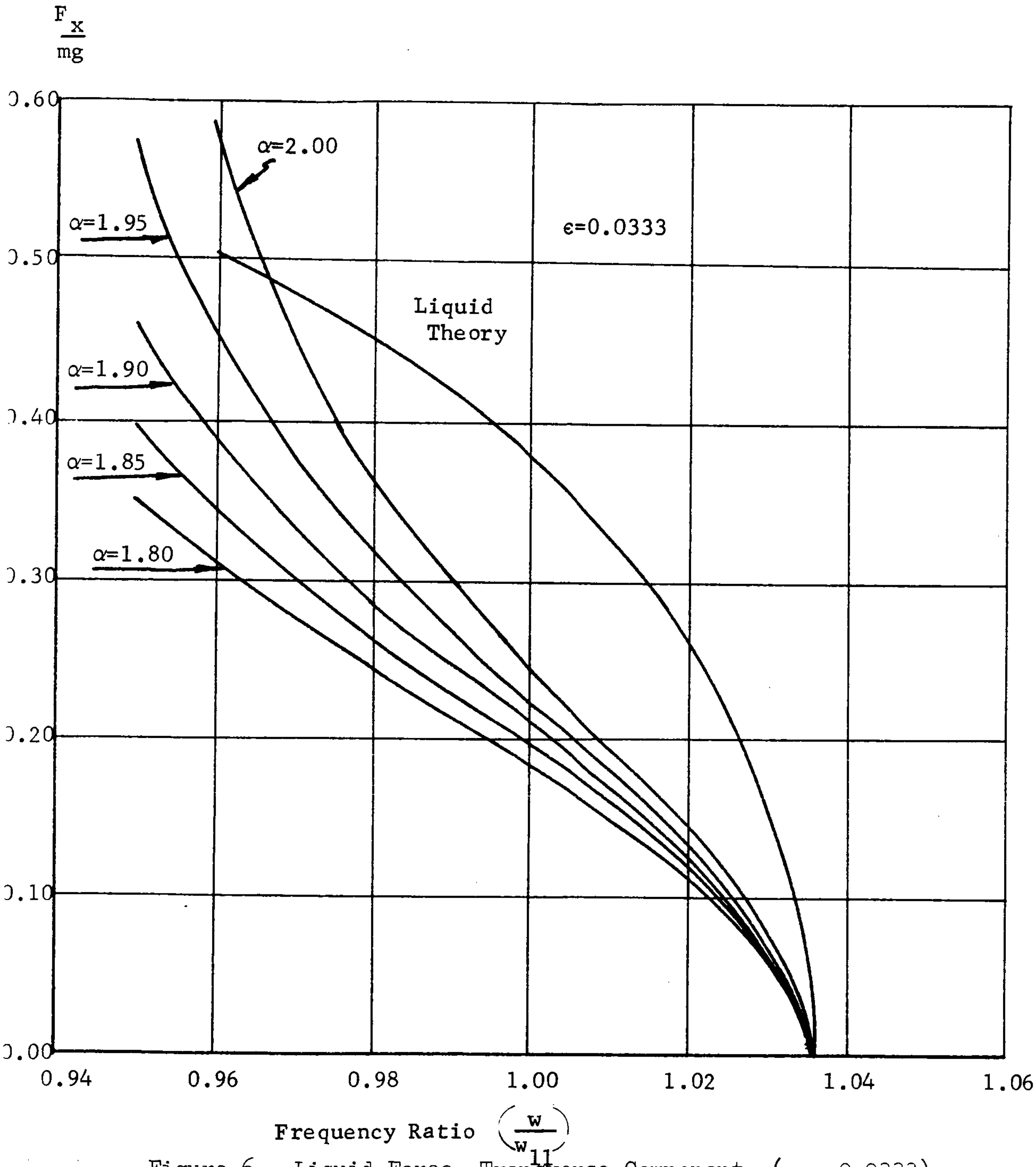
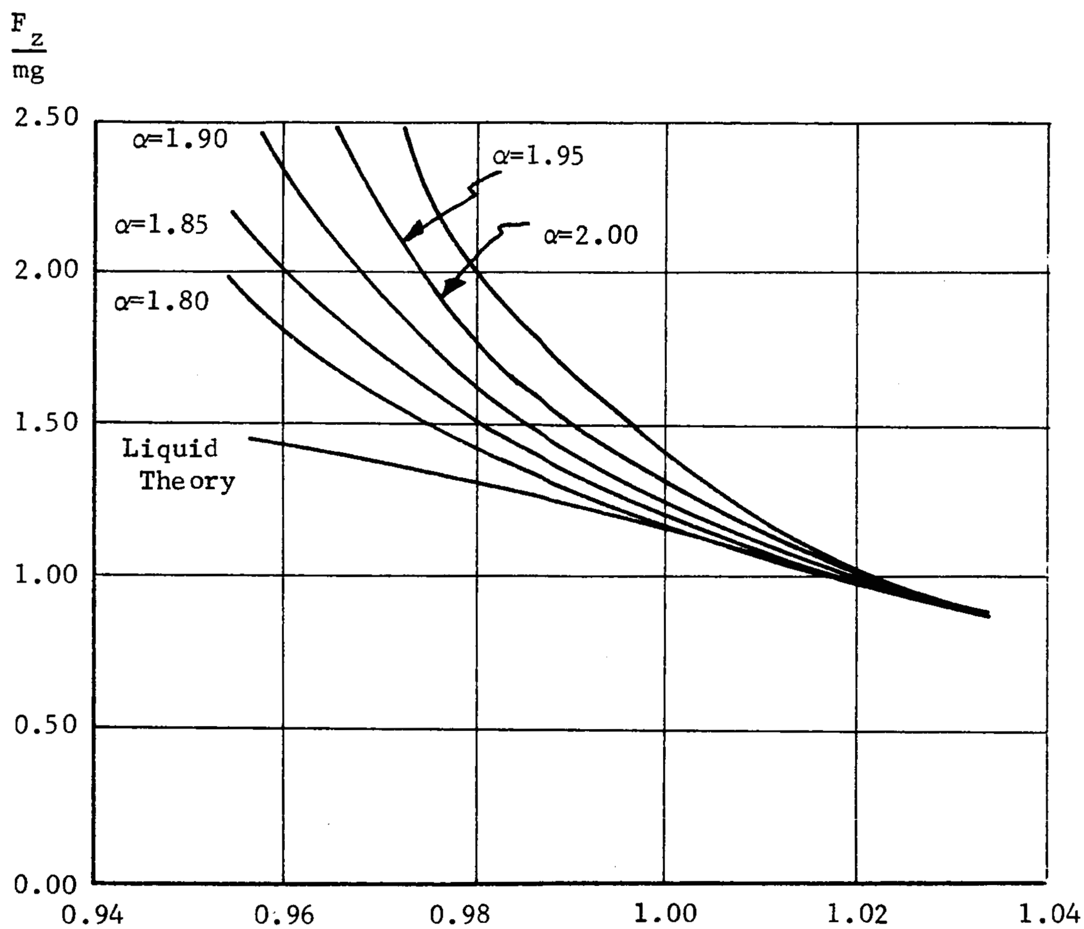


Figure 6. Liquid Force, Transverse Component ($\epsilon = 0.0333$)



Frequency Ratio $\left(\sigma = \frac{w}{w_{11}}\right)$

Figure 7. Liquid Force, Axial Component ($\epsilon = 0.0333$)

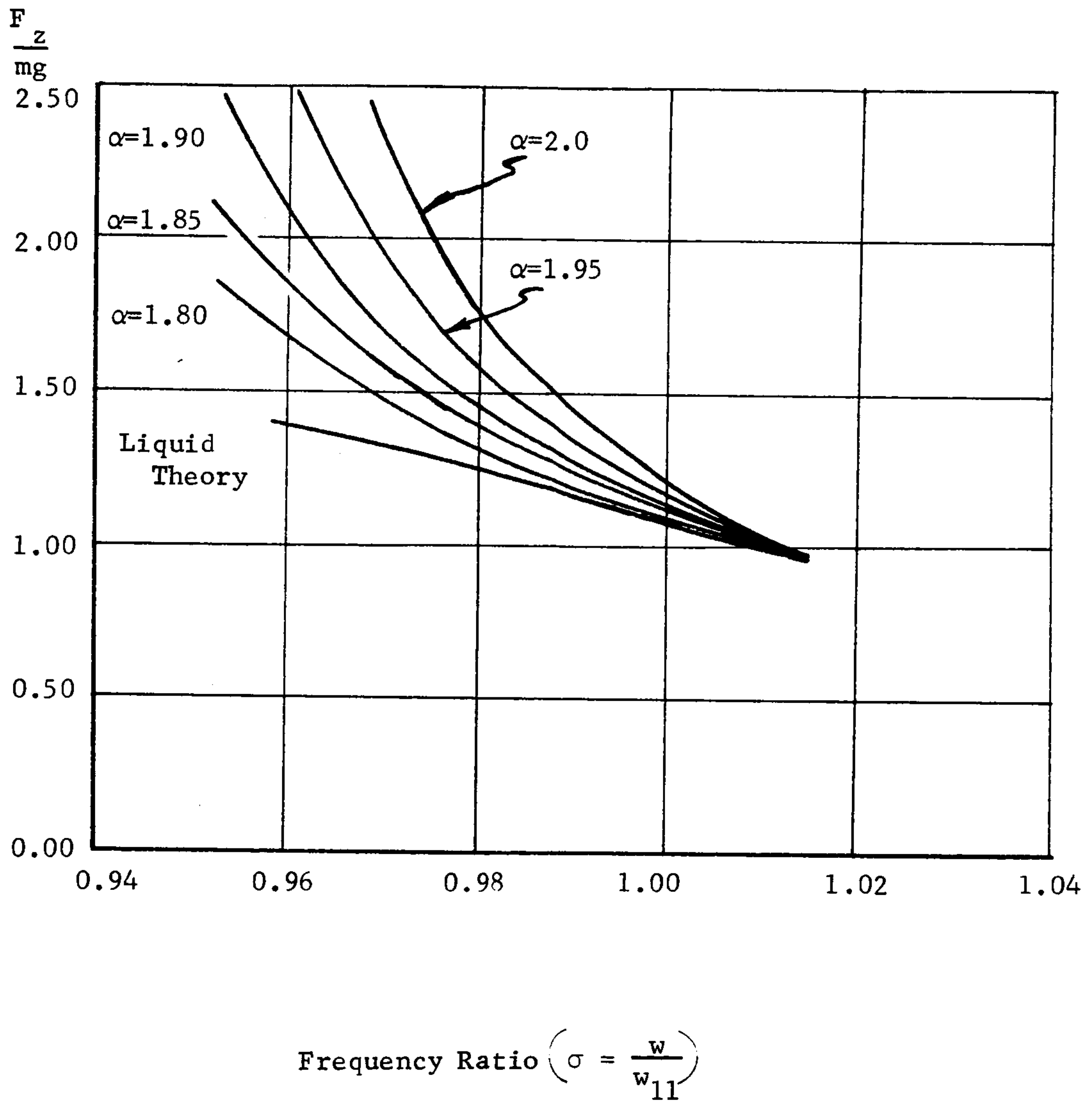


Figure 8. Liquid Force, Axial Component (ϵ = 0.0165)

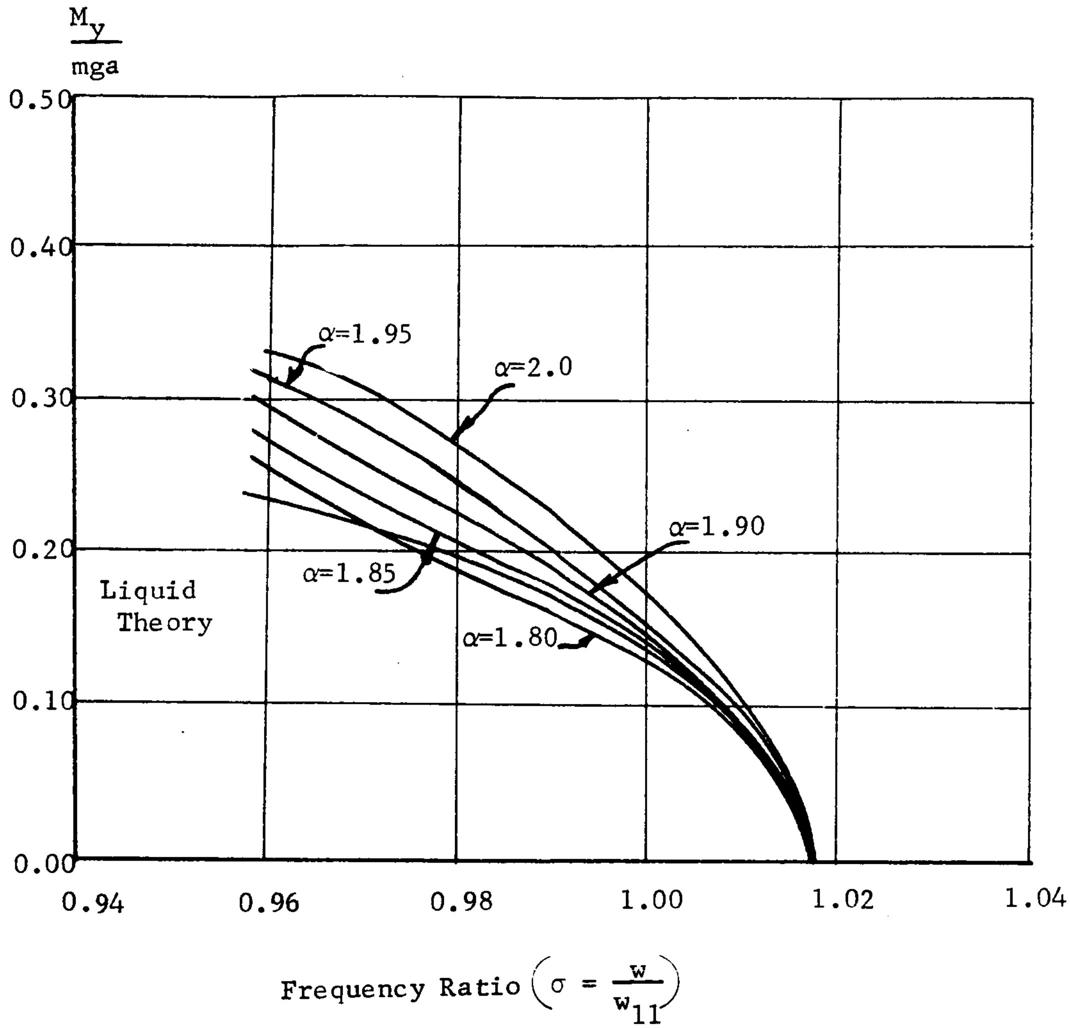
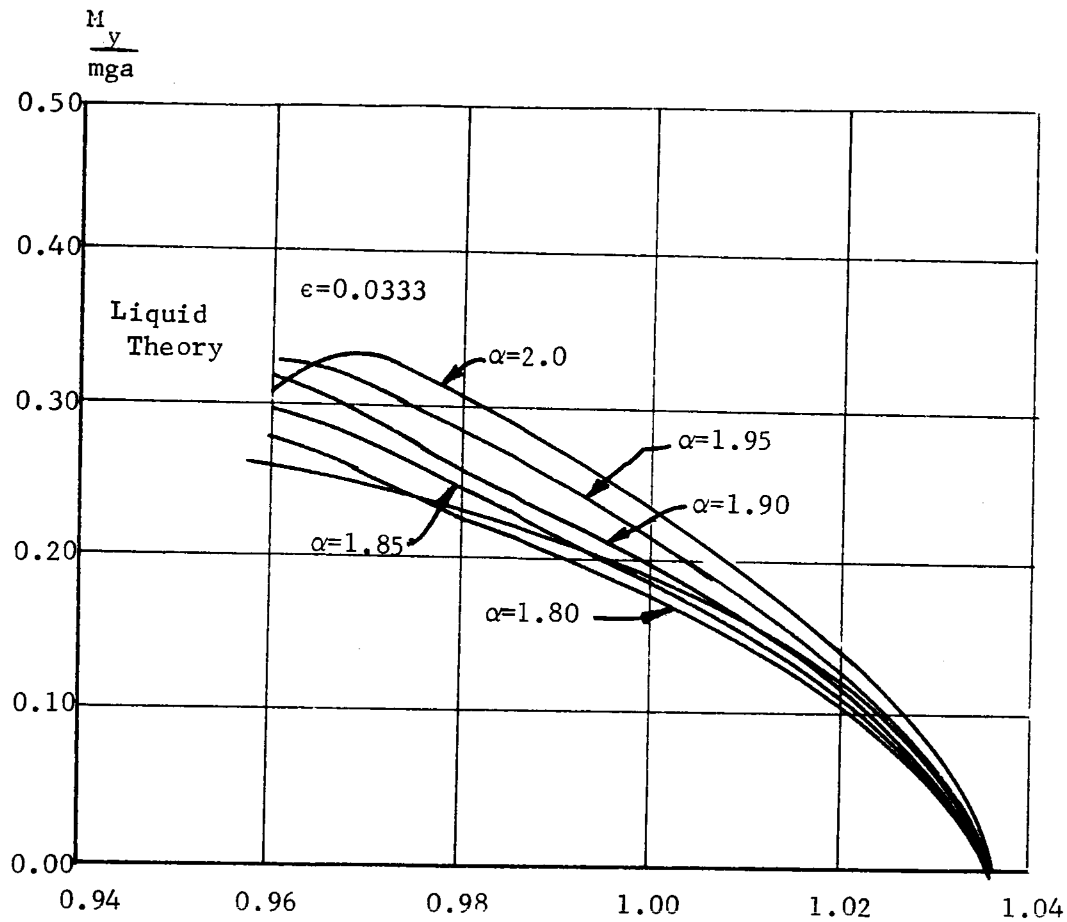


Figure 9. Liquid Moment Versus Frequency Parameter (ϵ = 0.0333)



Frequency Ratio $\left(\sigma = \frac{w}{w_{11}}\right)$

Figure 10. Liquid Moment Versus Frequency Parameter (ϵ = 0.0165)

6. REFERENCES

[1] Faraday, M., "On the Forms and States Assumed by Fluids in Contact with Vibrating Elastic Surfaces," Phil. Trans. Roy. Soc. 121, (1831), 319-340.

[2] Rayleigh, L., "On the Maintenance of Vibrations by Forces of Double Frequency, and on the Propagation of Waves through a Medium Endowed with a Periodic Structure," Phil. Mag., 24, (1889), 145-159.

[3] Dodge, Kana, and Abramson, "Liquid Surface Oscillations in Longitudinally Excited Rigid Cylindrical Containers," AIAA Journal, 3, No. 4, (April 1965), 685-695.

[4] Bauer, H. F., C. Clark, and J. Woodward, "Analytical Model for the Description of the Rotary Sloshing Motion," NASA Report, Contract No. NAS-8-11159, (May 1965).

[5] Lamb, Sir Horace, Hydrodynamics, Dover Publications, Sixth Edition, (1932), 6.